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Theoretical Increase About Interpolation Involving Finite Differences

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Abstract: *When we know the values of a function whose abscissa are equally spaced, we can utilize the traditional method, namely, that of Gregory-Newton, in order that we can determine the polynomial interpolation. However, in this paper we present an alternative technique .Will be seen, for example, under special conditions for sequences defined of recurrent form, in arithmetic progressions of higher order, which are performed only* $n^2 - n$ *operations.*

Keywords: Arithmetic Progressions, Finite Differences, Gregory-Newton , Polynomial Interpolation.

Resumo: *Quando se conhece os valores de uma função cujas abscissas sejam igualmente espaçadas, podemos utilizar o método tradicional, a saber, o de Gregory-Newton, a fim de que determinemos a interpolação polinomial. Entretanto neste artigo apresentaremos uma técnica alternativa. Será visto, por exemplo, em condições especiais de sequências definidas na forma recorrente, em progressões aritméticas de ordem superior, que são executadas apenas* $n^2 - n$ *operações.*

Palavras-chave : Progressões Aritméticas , Diferenças Finitas, Gregory-Newton, Interpolação Polinomial.

Introduction

The theory of polynomial interpolation is the foundation of several algorithms, among which the quadrature, the calculation of the roots of equations, the differentiation and the integration of differential equations. Nevertheless, it has its own large area, and tangentiating it there are other sections, such as the study of sequences called arithmetic progressions of higher order, which are necessarily determined by a polynomial function.

Generally the sequences are obtained through data collected from an experiment or a phenomenon, however there are problems in which the values of the abscissa are equally spaced. Aiming at its resolution, will be reported two formulas, the first to obtain the polynomial interpolating from a sequence of forward differences whose order is greater then to another that we knows the function. The second from a sequence of forward differences whose order is smaller than the other one.

Development

Firstly, let h be the fixed value of the abscissa spacing and be also the following polynomial of degree n

$$P_n(x) = a_{(n,n)} \cdot x^n + a_{(n-1,n)} \cdot x^{n-1} \cdots\ a_{(1,n)} \cdot x + a_{(0,n)}\ .$$

According to Lemma in (BARBOSA; BELLOMO; FILHO, 1973). The imposition of the operator $\triangle$ to a polynomial lowers the degree and the coefficient of the unknown with higher power is $h \cdot n \cdot a_{(n,n)}$. But we will present a more specific and analytical notation than the operator $\triangle$ and a more general theorem to the presented Lemma .

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Definition 0.1 *Let $(x_i, \triangle_h^w f(x_i)), 0 \leq i \leq m-w$, be $m+1-w$ distinct points and the values of x_i equally spaced such that $x_{i+1} - x_i = h, \forall i$.Also let $P_{m-w}(x)$ be the polynomial interpolating of degree $m-w$ of these $m-w+1$ points.*

1. *Indicates per $\bigvee_h^t P_{m-w}(x)$ the polynomial of degree $m-w-t$ which contains the points formed by forward differences of order $w+t$.*
2. *Indicates per $\bigwedge_h^t P_{m-w}(x)$ the polynomial of degree $m-w+t$ which contains the points formed by forward differences of order $w-t$.*

Theorem 0.1 *Let $(x_i, \triangle_h^w f(x_i)), 0 \leq i \leq n$, be $n+1$ distinct points and the values of x_i equally spaced such that $x_{i+1} - x_i = h, \forall i$.Also let $P_n(x)$ be the polynomial interpolating of degree n of these $n+1$ points.Then*

$$\bigvee_h^1 P_n(x) = \sum_{k=0}^{n-1} x^k \cdot \left(\sum_{j=k+1}^{n} h^{j-k} \cdot \binom{j}{k} \cdot a_{(j,n)} \right) \tag{1}$$

Proof. By definition in 2.1 $\bigvee_h^1 P_n(x_i) = \triangle_h^{w+1} f(x_i)$. Therefore

$$\bigvee_h^1 P_n(x_i) = \triangle_h^w f(x_i + h) - \triangle_h^w f(x_i) \tag{2}$$

By hypothesis $P_n(x_i) = \triangle_h^w f(x_i), 0 \leq i \leq n$ and by equation (2).So

$$\bigvee_h^1 P_n(x_i) = P_n(x_i + h) - P_n(x_i) \tag{3}$$

$$\Rightarrow \bigvee_h^1 P_n(x_i) = \sum_{j=0}^{n} (x_i + h)^j \cdot a_{(j,n)} - \sum_{j=0}^{n} x_i^j \cdot a_{(j,n)}$$

$$\Rightarrow \bigvee_h^1 P_n(x_i) = \sum_{j=0}^{n} ((x_i + h)^j - x_i^j) \cdot a_{(j,n)}$$

$$\Rightarrow \bigvee_h^1 P_n(x_i) = h \cdot \binom{1}{1} \cdot a_{(1,n)} + \left(h \cdot \binom{2}{1} \cdot x_i + h^2 \cdot \binom{2}{2} \right) \cdot a_{(2,n)} + \cdots +$$

$$\left(h \cdot \binom{n}{1} \cdot x_i^{n-1} + h^2 \cdot \binom{n}{2} \cdot x_i^{n-2} + \cdots + h^n \cdot \binom{n}{n} \right) \cdot a_{(n,n)}$$

$$\Rightarrow \bigvee_h^1 P_n(x_i) = h \cdot \binom{n}{1} \cdot a_{(n,n)} \cdot x_i^{n-1} +$$

$$\left(h^2 \cdot \binom{n}{2} \cdot a_{(n,n)} + h \cdot \binom{n-1}{1} \cdot a_{(n-1,n)} \right) \cdot x_i^{n-2} + \cdots +$$

$$\left(h^n \cdot \binom{n}{n} \cdot a_{(n,n)} + h^{n-1} \cdot \binom{n-1}{n-1} \cdot a_{(n-1,n)} + \cdots + h \cdot \binom{1}{1} \cdot a_{(1,n)} \right) \cdot x_i^0$$

$$\Rightarrow \bigvee_h^1 P_n(x_i) = h \cdot \binom{n}{n-1} \cdot a_{(n,n)} \cdot x_i^{n-1} +$$

$\left(h^2 \cdot \binom{n}{n-2} \cdot a_{(n,n)} + h \cdot \binom{n-1}{n-2} \cdot a_{(n-1,n)}\right) \cdot x_i^{n-2} + \cdots +$
$\left(h^n \cdot \binom{n}{0} \cdot a_{(n,n)} + h^{n-1} \cdot \binom{n-1}{0} \cdot a_{(n-1,n)} + \cdots + h \cdot \binom{1}{0} \cdot a_{(1,n)}\right) \cdot x_i^0$

For the Fundamental Theorem in (BARBOSA, 1973) there exists a single polynomial of degree no greater than $n-1$ satisfying these n point : $(x_i, \triangle_h^{w+1} f(x_i))$,
$0 \leq i \leq n-1$. So

$$\bigvee_h^1 P_n(x) = h \cdot \binom{n}{n-1} \cdot a_{(n,n)} \cdot x^{n-1} +$$
$$\left(h^2 \cdot \binom{n}{n-2} \cdot a_{(n,n)} + h \cdot \binom{n-1}{n-2} \cdot a_{(n-1,n)}\right) \cdot x^{n-2} + \cdots +$$
$$\left(h^n \cdot \binom{n}{0} \cdot a_{(n,n)} + h^{n-1} \cdot \binom{n-1}{0} \cdot a_{(n-1,n)} + \cdots + h \cdot \binom{1}{0} \cdot a_{(1,n)}\right) \cdot x^0$$
$$\Rightarrow \bigvee_h^1 P_n(x) = \left(\sum_{j=1}^{n} h^j \cdot \binom{j}{0} \cdot a_{(j,n)}\right) \cdot x^0 +$$
$$\left(\sum_{j=2}^{n} h^{j-1} \cdot \binom{j}{1} \cdot a_{(j,n)}\right) \cdot x^1 + \cdots + h \cdot \binom{n}{n-1} \cdot a_{(n,n)} \cdot x^{n-1}$$

Corollary 0.1 *Let $(x_i, \triangle_h^w f(x_i)), 0 \leq i \leq n$, be $n+1$ distinct points, and the values of x_i equally spaced such that $x_{i+1} - x_i = h, \forall i$ .Also let $P_n(x)$ be the polynomial interpolating of degree n of these $n+1$ points.Then*

$$\bigwedge_h^1 P_n(x) = a_{(0,n+1)} + \sum_{k=0}^{n} x^{k+1} \cdot \left(\frac{a_{(k,n)} - \displaystyle\sum_{j=k+2}^{n+1} h^{j-k} \cdot \binom{j}{k} \cdot a_{(j,n+1)}}{h \cdot \binom{k+1}{k}}\right) \tag{4}$$

Proof. From equation (1) we have that

$$a_{(k,n)} = \sum_{j=k+1}^{n+1} h^{j-k} \cdot \binom{j}{k} \cdot a_{(j,n+1)} \tag{5}$$

Then

$$a_{(k,n)} = h \cdot \binom{k+1}{k} \cdot a_{(k+1,n+1)} + h^2 \cdot \binom{k+2}{k} \cdot a_{(k+2,n+1)} + \cdots +$$
$$h^{n+1-k} \cdot \binom{n+1}{k} \cdot a_{(n+1,n+1)}$$
$\Rightarrow$

$$a_{(k+1,n+1)} = \left(\frac{a_{(k,n)} - \displaystyle\sum_{j=k+2}^{n+1} h^{j-k} \cdot \binom{j}{k} \cdot a_{(j,n+1)}}{h \cdot \binom{k+1}{k}}\right) \tag{6}$$

Applications

Here, to present the efficacy of the described models in the areas of arithmetic progressions of higher order and polynomial Interpolation we apply some examples.

Example 0.1 *Given the pairs* $(0;1),(0,5;0,375),(1;0)$.*Establish the polynomial interpolating* $P_2(x)$.

Tabela 1: Difference Table

i	0	1	2
x_i	0	$0,5$	1
y_i	1	$0,375$	0
$\triangle^1_{0,5}y_i$	$-0,625$	$-0,375$	
$\triangle^2_{0,5}y_i$	$0,25$		

$$\Rightarrow \bigwedge_{0,5}^{1} P_0(x) = \frac{\triangle^2_{0,5}y_0}{h\cdot\binom{1}{0}}\cdot x + \triangle^1_{0,5}y_0 = 0,5x - 0,625$$

$$\Rightarrow \bigwedge_{0,5}^{1} P_1(x) = \frac{a_{(1,1)}}{h\cdot\binom{2}{1}}\cdot x^2 + \frac{a_{(0,1)} - h^2\cdot\binom{2}{0}\cdot a_{(2,2)}}{h\cdot\binom{1}{0}}\cdot x + \triangle^1_{0,5}y_0 = \frac{x^2}{2} - \frac{3x}{2} + 1.$$

Example 0.2 *Establish the general term of the sequence* $\{a_n\}$ *defined by the form:*

$$\begin{cases} a_1 = 11 \\ a_{n+1} = a_n + 2n^3 + 3n^2 + 5n + 7 \end{cases}$$

We have:

$\triangle a_n = 2n^3 + 3n^2 + 5n + 7$

$\Rightarrow a_n = \bigwedge_{1}^{1} 2n^3 + 3n^2 + 5n + 7$.

Applying equation (6) recursively :

$$\begin{cases} a_{(4,4)} = \frac{2}{4} = \frac{1}{2} \\ a_{(3,4)} = \left(3 - 6\cdot\frac{1}{2}\right)\cdot\frac{1}{3} = 0 \\ a_{(2,4)} = \left[5 - \left(3\cdot 0 + 4\cdot\frac{1}{2}\right)\right]\cdot\frac{1}{2} = \frac{3}{2} \\ a_{(1,4)} = 7 - \left(\frac{3}{2} + 0 + \frac{1}{2}\right) = 5 \end{cases}$$

And $a_{(0,4)} = a_0 = a_1 - \triangle a_0 = 11 - 7 = 4$

$\Rightarrow a_n = \bigwedge_{1}^{1} 2n^3 + 3n^2 + 5n + 7 = \frac{n^4}{2} + \frac{3n^2}{2} + 5n + 4$.

Example 0.3 *Establish the sum of a sequence given by the general term* $\{n^3\}$.

Demonstrates that $\triangle S_n = a_{n+1}$ (BARBOSA, 1973) . So :

$\triangle S_n = (n+1)^3 = n^3 + 3n^2 + 3n + 1$
$\Rightarrow S_n = \bigwedge_1^1 n^3 + 3n^2 + 3n + 1$.

Applying equation (6) recursively :

$$\begin{cases} a_{(4,4)} = \dfrac{1}{4} \\ a_{(3,4)} = \left(3 - 6 \cdot \dfrac{1}{4}\right) \cdot \dfrac{1}{3} = \dfrac{1}{2} \\ a_{(2,4)} = \left[3 - \left(3 \cdot \dfrac{1}{2} + 4 \cdot \dfrac{1}{4}\right)\right] \cdot \dfrac{1}{2} = \dfrac{1}{4} \\ a_{(1,4)} = 1 - \left(\dfrac{1}{4} + \dfrac{1}{2} + \dfrac{1}{4}\right) = 0 \end{cases}$$

As $S_0 = S_1 - \triangle S_0, S_1 = a_1$ and $\triangle S_0 = a_1$. Then $a_{(0,4)} = 0$.
$\Rightarrow S_n = \bigwedge_1^1 n^3 + 3n^2 + 3n + 1 = \dfrac{n^4}{4} + \dfrac{n^3}{2} + \dfrac{n^2}{4}$.

Final considerations

If we not consider the number of operations to obtain binomial numbers, the calculation of the powers of spacing and the determination of $a_{(0,n)}$, then for the application from equation (4) in the calculation of $\bigwedge_h^1 P_{n-1}(x), n \geq 1$, were done n divisions, $\dfrac{n^2}{2} - \dfrac{n}{2}$ additions and $n^2 - n$ products, totalizing $\dfrac{3n^2 - n}{2}$ operations.Now for spacing equal to one, the calculation performed reduces to $n^2 - n$.

In this way, it is possible to observe the efficiency in the use of the technique, when comparing to other methods adopted in the literature, both to obtain the general term, in which the sequence is exposed of inductive manner, and to get the polynomial from the sum of a arithmetic sequence of order k, declared analytically.

References

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