

Certain integrals of generalized hypergeometric and confluent hypergeometric functions

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Abstract: *In this paper, we aim at establishing certain finite integral formulas for the generalized Gauss hypergeometric and confluent hypergeometric functions. Furthermore, the $F_p^{(\alpha, \beta)}(a, b; c; z)$ -function occurring in each of our main results can be reduced, under various special cases, to such simpler functions as the classical Gauss hypergeometric function ${}_2F_1$, Gauss confluent hypergeometric function $\varphi_p^{(\alpha, \beta)}(b; c; z)$ function and generalized hypergeometric function ${}_pF_q$. A specimen of some of these interesting applications of our main integral formulas are presented briefly.*

Keywords: Generalized Beta function; Generalized Gamma functions; Generalized Gauss hypergeometric functions; Confluent hypergeometric functions

Introduction and Preliminaries

In many areas of applied mathematics, various types of special functions become essential tools for scientists and engineers. The continuous development of mathematical physics, probability theory and other areas has led to new classes of special functions and their extensions and generalizations (see, for details, [Srivastava, Chaudhry and Agarwal \(2012\)](#) and the references cited therein; see also [Srivastava \(2011\)](#), [Srivastava and Choi \(2001\)](#), [Srivastava and Choi \(2012\)](#)).

A lot of research work has recently come up on the study and development of the functions, which are more general than the Beta type function $\beta(x, y)$, popularly known as generalized Beta type functions. These functions, as a part of the theory of confluent hypergeometric functions, are important special functions and their closely related ones are widely used in physics and engineering. Moreover, generalized Beta functions ([CHAUDHRY et al., 1997](#); [CHAUDHRY et al., 2004](#)) have played a pivotal role in the advancement of further research and have proved to be exemplary in nature. The Euler's gamma function $\Gamma(z)$ is one of the most fundamental special functions, because of its important role in various fields in the mathematical, physical, engineering and statistical sciences. Various generalizations of the gamma function can be found in the literature ([CHAUDHRY; ZUBAIR, 1994](#); [GUSEINOV; MAMEDOV, 2007](#); [KATAYAMA, 2010](#); [KOBAYASHI, 1991](#); [SRIVASTAVA; SAXENA; RAM, 2005](#)).

The following extension of the gamma function is introduced by [Chaudhry and Zubair \(1994\)](#):

$$\Gamma_p(z) = \begin{cases} \int_0^\infty t^{z-1} \exp\left(-t - \frac{p}{t}\right) dt, & \Re(p) > 0; z \in \mathbb{C} \\ \Gamma(z), & \Re(z) > 0 \end{cases} \quad (1)$$

The extension of Euler's beta function is considered by [Chaudhry et al. \(1997\)](#) in the following form:

$$\beta_p(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left(\frac{-p}{t(1-t)}\right) dt, \quad \Re(p) > 0, \Re(x) > 0, \Re(y) > 0. \quad (2)$$

[Chaudhry et al. \(2004\)](#) used $\beta_p(x, y)$ to extend the hypergeometric function, known as the extended Gauss hypergeometric function, as follows:

$$F_p(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{\beta_p(b+n, c-b)}{\beta(b, c-b)} \frac{(z)^n}{n!}, \quad p \geq 0, \Re(c) > \Re(b) > 0, \quad (3)$$

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where $(a)_n$ denotes the Pochhammer symbol defined as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n=0; a \in \mathbb{C}/\{0\} \\ a(a+1)(a+2)\dots(a+n-1), & n \in \mathbb{N}, a \in \mathbb{C}. \end{cases}$$

The integral representation of Euler's type function is

$$F_p(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \exp\left(\frac{-p}{t(1-t)}\right) dt, \quad (4)$$

where $p \geq 0$ and $|\arg(1-z)| < \pi < p; \Re(c) > \Re(b) > 0$.

Also, the extended confluent hypergeometric function is defined as

$$\varphi_p(b; c; z) = \sum_{n=0}^{\infty} \frac{\beta_p(b+n, c-b)}{\beta(b, c-b)} \frac{z^n}{n!}, \quad p \geq 0, \Re(c) > \Re(b) > 0. \quad (5)$$

The transformation formulas, recurrence relations, summation and asymptotic formulas, differentiation properties, the Mellin transforms and some new representations of these extended functions can be found in many earlier work (CHAUDHRY et al., 2004; ÖZARASLAN, 2012; ÖZERGİN; ÖZARASLAN; ALTIN, 2011; SRIVASTAVA, 2005; SRIVASTAVA; CHOI, 2012).

The generalized Euler's gamma function is defined in Özergin, Özaraslan and Altin (2011) as

$$\Gamma_p^{(\alpha, \beta)}(x) = \int_0^{\infty} t^{x-1} {}_1F_1\left(\alpha; \beta; -t - \frac{p}{t}\right) dt, \quad \Re(\alpha) > 0, \Re(\beta) > 0, \Re(p) > 0, \Re(x) > 0. \quad (6)$$

Recently, Özergin (2011) introduced and studied some fundamental properties and characteristics of the generalized Beta type function $\beta_p^{(\alpha, \beta)}(x, y)$ in their Ph.D. Thesis and defined by (see, e.g., Özergin (2011, p. 32)):

$$\beta_p^{(\alpha, \beta)}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) dt, \quad (7)$$

with $\Re(p) \geq 0, \min(\Re(x), \Re(y), \Re(\alpha), \Re(\beta)) > 0$ and $\beta_0^{(\alpha, \beta)}(x, y) = \beta(x, y)$, where $\beta(x, y)$ is a well known Euler's Beta function defined by

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad ((\Re(x)) > 0, (\Re(y)) > 0). \quad (8)$$

Similarly, by appealing to $\beta_p^{\alpha, \alpha}(x, y)$, Özergin et al. introduced and investigated a further extension of the following potentially useful generalized Gauss hypergeometric functions defined as follows (see, e.g., Özergin, Özaraslan and Altin (2011, p. 4606, Sec. 3); see also Özergin (2011, p. 39, Ch. 4)):

$$F_p^{(\alpha, \beta)}(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{z^n}{n!}, \quad (|z| < 1), \quad (9)$$

and

$${}_1F_1^{(\alpha, \beta; p)}(b; c; z) = \sum_{n=0}^{\infty} \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{z^n}{n!}, \quad (|z| < 1), \quad (10)$$

corresponding integral representations are given by Özergin, Özaraslan and Altin (2011):

$$F_p^{(\alpha, \beta)}(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) (1-zt)^{-a} dt, \quad (11)$$

for $\Re(p) \geq 0$, and $|\arg(1-z)| < \pi < p$; $\Re(c) > \Re(b) > 0$.

It is obvious to see that (CHAUDHRY et al., 2004):

$$F_p^{(\alpha, \alpha)}(a, b; c; z) = F_p(a, b; c; z), \quad F_0^{(\alpha, \beta)}(a, b; c; z) = {}_2F_1(a, b; c; z);$$

$${}_1F_1^{(\alpha, \alpha; p)}(b; c; z) = {}_1F_1^p(b; c; z) = \varphi_p(b; c; z), \quad {}_1F_1^{(\alpha, \beta; 0)}(b; c; z) = {}_1F_1(b; c; z). \quad (12)$$

where the ${}_2F_1(\cdot)$ is a special case of the well-known generalized hypergeometric series ${}_pF_q(\cdot)$ defined by (see, e.g., Srivastava and Choi (2012, Sec. 1.5); see also Srivastava and Karlsson (1985)).

$${}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} = {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z), \quad (13)$$

where $z \in \mathbb{C}, p \leq q, \alpha_i, \beta_j \in \mathbb{C}, \beta_j \neq 0, -1, -2, \dots, (i = 1, 2, \dots, p, j = 1, 2, \dots, q)$.

Integrals involving generalized Gauss hypergeometric and Confluent hypergeometric function

In this section we calculate the $F_p^{(\alpha, \beta)}(a, b; c; z)$ -function with some algebraic function.

Theorem 1. For the generalized Gauss hypergeometric function, we have the following integral

$$\int_0^1 x^{-\rho} (1-x)^{\rho-\sigma-1} F_p^{(\alpha, \beta)}(a, b; c; kx) dx$$

$$= \beta(1-\rho, \rho-\sigma) \sum_{n=0}^{\infty} \frac{(a)_n (1-\rho)_n}{(1-\sigma)_n} \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{k^n}{n!}, \quad (14)$$

where, $\Re(p) \geq 0$, and $|\arg(1-kx)| < \pi < p$; $\Re(c) > \Re(b) > 0$

Proof. Making use of relation (9), it gives

$$\mathcal{I}_1 = \int_0^1 x^{-\rho} (1-x)^{\rho-\sigma-1} \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{(kx)^n}{n!} dx$$

$$= \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{(k)^n}{n!} \int_0^1 x^{n-\rho} (1-x)^{\rho-\sigma-1} dx$$

$$= \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{(k)^n}{n!} \beta(n-\rho+1, \rho-\sigma)$$

$$= \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{(k)^n}{n!} \frac{\Gamma(n-\rho+1) \Gamma(\rho-\sigma)}{\Gamma(n-\sigma+1)}$$

$$= \frac{\Gamma(1-\rho) \Gamma(\rho-\sigma)}{\Gamma(1-\sigma)} \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{(k)^n}{n!} \frac{(1-\rho)_n}{(1-\sigma)_n}$$

$$= \beta(1-\rho, \rho-\sigma) \sum_{n=0}^{\infty} \frac{(a)_n (1-\rho)_n}{(1-\sigma)_n} \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{k^n}{n!}.$$

This complete the proof of the Theorem 1. □

If we set $p = 0$ in above result then we obtain the special case of (14) in terms of classical Gauss hypergeometric function as given in the following result:

Corollary 2.

$$\int_0^1 x^{-\rho} (1-x)^{\rho-\sigma-1} {}_2F_1(a, b; c; kx) dx = \beta(1-\rho, \rho-\sigma) {}_3F_2\left(\begin{matrix} a, b, 1-\rho \\ 1-\sigma \end{matrix}; k\right). \quad (15)$$

The integral of Gauss Confluent hypergeometric function is given by

Corollary 3.

$$\begin{aligned} & \int_0^1 x^{-\rho} (1-x)^{\rho-\sigma-1} \varphi_p^{(\alpha, \beta)}(b; c; kx) dx \\ &= \sum_{n=0}^{\infty} \beta(1+n-\rho, \rho-\sigma) \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b) (k)^n}{\beta(b, c-b) n!}. \end{aligned} \quad (16)$$

Moreover, for the generalized hypergeometric function ${}_pF_q$, we have the following corollary:

Corollary 4.

$$\begin{aligned} & \int_0^1 x^{-\rho} (1-x)^{\rho-\sigma-1} {}_pF_q\left[\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix}; kx\right] dx \\ &= \beta(1-\rho, \rho-\sigma) {}_{p+1}F_{q+1}\left[\begin{matrix} \alpha_1, \dots, \alpha_p, 1-\rho \\ \beta_1, \dots, \beta_q, 1-\sigma \end{matrix}; k\right] \end{aligned} \quad (17)$$

for $x \in \mathbb{C}, p \leq q; \alpha_j, \beta_j \in \mathbb{C}, \beta \neq 0, -1, -2, \dots; i = (\overline{1, p}), j = (\overline{1, q})$.

Theorem 5.

$$\begin{aligned} & \int_1^{\infty} x^{-\rho} (x-1)^{\sigma-1} F_p^{(\alpha, \beta)}(a, b; c; kx) dx \\ &= \beta(\sigma, \rho-\sigma) \sum_{n=0}^{\infty} \frac{(a)_n (\rho)_{-n} \beta_p^{(\alpha, \beta)}(b+n, c-b) k^n}{(\rho-\sigma)_{-n} \beta(b, c-b) n!}. \end{aligned} \quad (18)$$

Proof.

$$\begin{aligned} \mathcal{I}_2 &= \int_1^{\infty} x^{-\rho} (x-1)^{\sigma-1} \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b) (kx)^n}{\beta(b, c-b) n!} dx \\ &= \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b) (k)^n}{\beta(b, c-b) n!} \int_1^{\infty} x^{-\rho+n} (x-1)^{\sigma-1} dx. \end{aligned}$$

Let $x = 1 + t$, then we arrive at

$$\mathcal{I}_2 = \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b) (k)^n}{\beta(b, c-b) n!} \int_0^{\infty} t^{\sigma-1} (1+t)^{-\rho+n} dt,$$

using the formula

$$\Gamma(\alpha) \Gamma(\beta) = \Gamma(\alpha + \beta) \int_0^{\infty} x^{\alpha-1} (1+x)^{-(\alpha+\beta)} dx, \quad (19)$$

then we have

$$\begin{aligned} \mathcal{I}_2 &= \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b) (k)^n}{\beta(b, c-b) n!} \frac{\Gamma(\sigma) \Gamma(\rho-\sigma-n)}{\Gamma(\rho-n)} \\ &= \beta(\sigma, \rho-\sigma) \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b) (k)^n}{\beta(b, c-b) n!} \frac{(\rho)_{-n}}{(\rho-\sigma)_{-n}}. \end{aligned}$$

This complete the proof of the Theorem 5. □

We can also obtain special cases of Theorem 5 as done in Theorem 1.

Theorem 6.

$$\begin{aligned} & \int_0^{\infty} e^{-kx} x^{\rho-1} F_p^{(\alpha, \beta)}(a, b; c; lx) dx \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\rho)}{k^\rho} (a)_n (\rho)_n \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b)}{n! \beta(b, c-b)} \left(\frac{l}{k}\right)^n, \quad k \neq 0. \end{aligned} \quad (20)$$

Proof. Using (11) and $(1-lxt)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} (lxt)^n$, we have

$$\begin{aligned} \mathcal{I}_3 &= \frac{1}{\beta(b, c-b)} \int_0^{\infty} \int_0^1 t^{b-1} (1-t)^{c-b-1} e^{-kx} x^{\rho-1} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) \\ &\quad \times \sum_{n=0}^{\infty} \frac{(a)_n}{n!} (lxt)^n dx dt, \\ &= \frac{1}{\beta(b, c-b)} \sum_{n=0}^{\infty} (a)_n \frac{l^n}{n!} \int_0^{\infty} \int_0^1 t^{b+n-1} (1-t)^{c-b-1} e^{-kx} x^{\rho+n-1} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) dx dt \\ &= \sum_{n=0}^{\infty} \frac{(a)_n l^n}{\beta(b, c-b) n!} \int_0^{\infty} e^{-kx} x^{\rho+n-1} dx \int_0^1 t^{b+n-1} (1-t)^{c-b-1} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) dt \\ &= \sum_{n=0}^{\infty} \frac{(a)_n l^n}{\beta(b, c-b) n!} \int_0^{\infty} e^{-kx} x^{\rho+n-1} \beta_p^{(\alpha, \beta)}(b+n, c-b) dx \end{aligned} \quad (21)$$

because,

$$\frac{1}{k} \int_0^{\infty} e^{-\sigma} \left(\frac{\sigma}{k}\right)^{\rho+n-1} d\sigma = \frac{1}{k^{\rho+n}} \int_0^{\infty} e^{-\sigma} (\sigma)^{\rho+n-1} d\sigma$$

then we have

$$\mathcal{I}_3 = \sum_{n=0}^{\infty} \frac{(a)_n (\rho)_n \Gamma(\rho)}{k^\rho n!} \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b)}{\beta(b, c-b)} \left(\frac{l}{k}\right)^n.$$

This complete the proof of Theorem 6. □

If we set $p = 0$ in (20) then we obtain the following corollary:

Corollary 7.

$$\int_0^{\infty} e^{-kx} x^{\rho-1} {}_2F_1(a, b; c; lx) dx = \frac{\Gamma(\rho)}{k^\rho} {}_3F_2\left(\begin{matrix} a, b; \rho \\ c \end{matrix}; \frac{l}{k}\right), \quad k \neq 0. \quad (22)$$

The integral of Gauss Confluent hypergeometric function is given by the following result:

Corollary 8.

$$\begin{aligned} & \int_0^{\infty} e^{-kx} x^{\rho-1} \varphi_p^{(\alpha, \beta)}(b; c; lx) dx \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\rho+n)}{k^\rho n!} \frac{\beta_p^{(\alpha, \beta)}(b+n, c-b)}{\beta(b, c-b)} \left(\frac{l}{k}\right)^n, \quad k \neq 0. \end{aligned} \quad (23)$$

This result is in complete agreement with the result given in (ISMAIL, 2005, p.98).

Next, for the generalized hypergeometric function ${}_pF_q$, we have the following corollary:

Corollary 9. For $x \in \mathbb{C}, p \leq q; \alpha_j, \beta_j \in \mathbb{C}, \beta \neq 0, -1, -2, \dots; (i = (\overline{1, p}); j = (\overline{1, q}))$, we have

$$\int_0^\infty e^{-kx} x^{\rho-1} {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} \middle| lx \right] dx = \frac{\Gamma(\rho)}{k^\rho} {}_{p+1}F_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p, \rho \\ \beta_1, \dots, \beta_q \end{matrix} ; \frac{l}{k} \right], \quad k \neq 0. \tag{24}$$

Theorem 10.

$$\begin{aligned} & \int_0^\infty x^{\rho-1} (x + \beta)^{-\sigma} F_p^{(\delta, \eta)}(a, b; c; kx) dx \\ &= \beta(\rho, \sigma - \rho) \beta^{\rho+n-\sigma} \sum_{n=0}^\infty (a)_n (\rho)_n (\sigma - \rho)_{-n} \frac{\beta_p^{(\delta, \eta)}(b + n, c - b) k^n}{\beta(b, c - b) n!}. \end{aligned} \tag{25}$$

Proof.

$$\begin{aligned} \mathcal{I}_4 &= \int_0^\infty x^{\rho-1} (x + \beta)^{-\sigma} \sum_{n=0}^\infty (a)_n \frac{\beta_p^{(\delta, \eta)}(b + n, c - b) (kx)^n}{\beta(b, c - b) n!} dx \\ &= \sum_{n=0}^\infty (a)_n \frac{\beta_p^{(\delta, \eta)}(b + n, c - b) (k)^n}{\beta(b, c - b) n!} \int_0^\infty x^{\rho+n-1} (x + \beta)^{-\sigma} dx. \end{aligned}$$

Let $x = \beta t$, then we arrive at

$$\mathcal{I}_4 = \sum_{n=0}^\infty (a)_n \frac{\beta_p^{(\delta, \eta)}(b + n, c - b) (k)^n}{\beta(b, c - b) n!} \beta^{\rho-\sigma+n} \int_0^\infty t^{\rho+n-1} (1 + t)^{-\sigma} dt,$$

using the relation (19), then we have

$$\begin{aligned} \mathcal{I}_4 &= \sum_{n=0}^\infty (a)_n \frac{\beta_p^{(\delta, \eta)}(b + n, c - b) (k)^n}{\beta(b, c - b) n!} \beta^{\rho+n-\sigma} \frac{\Gamma(\sigma + n) \Gamma(\sigma - \rho - n)}{\Gamma(\sigma)} \\ &= \beta(\rho, \sigma - \rho) \sum_{n=0}^\infty \beta^{\rho+n-\sigma} (a)_n (\rho)_n (\sigma - \rho)_{-n} \frac{\beta_p^{(\delta, \eta)}(b + n, c - b) (k)^n}{\beta(b, c - b) n!}, \end{aligned}$$

then we easily get the R.H.S. of (25). This complete the proof of the Theorem 10. □

Theorem 11.

$$\begin{aligned} & \int_{-1}^1 (1 - x)^\rho (1 + x)^\sigma F_p^{(\alpha, \beta)}(a, b; c; kx) dx \\ &= 2^{\rho+\sigma+1} \beta(\rho + 1, \sigma + 1) \sum_{n=0}^\infty \frac{(a)_n \beta_p^{(\alpha, \beta)}(b + n, c - b)}{\beta(b, c - b) n!} F(-n, \rho + 1, \rho + \sigma + 2; 2) \frac{k^n}{n!}. \end{aligned} \tag{26}$$

Proof.

$$\begin{aligned} \mathcal{I}_5 &= \int_{-1}^1 (1 - x)^\rho (1 + x)^\sigma \sum_{n=0}^\infty (a)_n \frac{\beta_p^{(\alpha, \beta)}(b + n, c - b) (kx)^n}{\beta(b, c - b) n!} dx \\ &= \sum_{n=0}^\infty (a)_n \frac{\beta_p^{(\alpha, \beta)}(b + n, c - b) (k)^n}{\beta(b, c - b) n!} \int_{-1}^1 (1 - x)^\rho (1 + x)^\sigma x^n dx. \end{aligned}$$

Now, by putting $\frac{1-x}{2} = t \Rightarrow dx = -2 dt$, and using the integral representation of Gauss hypergeometric series

$$F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - tx)^{-a} dt, \tag{27}$$

then we arrive at the following result after a little simplification:

$$\mathcal{I}_5 = 2^{\rho+\sigma+1} \frac{\Gamma(\rho+1)\Gamma(\sigma+1)}{\Gamma(\rho+\sigma+2)} \sum_{n=0}^{\infty} \frac{(a)_n \beta_p^{(\alpha,\beta)}(b+n, c-b) (k)^n}{\beta(b, c-b) n!} \\ \times F(-n, \rho+1, \rho+\sigma+2; 2),$$

then we obtain the desired result in (26). This completes the proof. \square

The next theorem considers the behavior of the generalized Gauss hypergeometric function using the gamma function.

Theorem 12.

$$\lim_{\gamma \rightarrow -l} (\Gamma(\gamma))^{-1} F_p^{(\alpha,\beta)}(a, b; \gamma; x) = \frac{x^{l+1}}{\Gamma(a)\Gamma(b)\Gamma(-b-l)} \\ \times \sum_{r=0}^{\infty} \frac{(a+l+r)!}{(l+r+1)!} \beta_p^{(\alpha,\beta)}(b+l+r+1, -b-l) x^r. \quad (28)$$

Proof. Making use of (9), we have

$$\mathcal{I}_6 = \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha,\beta)}(b+n, \gamma-b) x^n}{\beta(b, \gamma-b) n!} \lim_{\gamma \rightarrow -l} \frac{1}{\Gamma(\gamma)},$$

by using (7), we obtain

$$\mathcal{I}_6 = \sum_{n=0}^{\infty} \frac{(a)_n}{\Gamma(b)} \int_0^1 t^{b+n-1} (1-t)^{-b-1} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) \frac{x^n (1-t)^{-l}}{n! \Gamma(-l-b)} dt,$$

by setting $n = l + r + 1$ and using (7), it yields

$$= \frac{(x)^{l+1}}{\Gamma(b)\Gamma(-l-b)} \sum_{r=0}^{\infty} \frac{(a)_{l+r+1}}{(l+r+1)!} \beta_p^{(\alpha,\beta)}(b+l+r+1, -b-l) x^r,$$

then we arrive at the desired result in (27). \square

The special cases of (28) for the generalized Gauss hypergeometric function and confluent hypergeometric function are given in the following corollaries:

Corollary 13.

$$\lim_{\gamma \rightarrow -l} (\Gamma(\gamma))^{-1} {}_2F_1(a, b; \gamma; x) = x^{l+1} \frac{(a)_{l+1} (b)_{l+1}}{(l+1)!} {}_2F_1\left[\begin{matrix} a+l+1, b+l+1 \\ l+2 \end{matrix}; x\right]. \quad (29)$$

Corollary 14.

$$\lim_{\gamma \rightarrow -l} (\Gamma(\gamma))^{-1} \varphi_p^{(\alpha,\beta)}(b; \gamma; x) = \frac{x^{l+1}}{\Gamma(-l)} \sum_{r=0}^{\infty} \frac{1}{\Gamma(l+r+2)} \frac{\beta_p^{(\alpha,\beta)}(b+l+r+1, -b-l)}{\beta(b, -l-b)} \frac{x^r}{r!}. \quad (30)$$

Integrals involving Gauss hypergeometric function with Jacobi Polynomials

The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ (RAINVILLE, 1960, p. 254) is defined as following:

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1+\alpha+\beta+n; \\ 1+\alpha; \end{matrix} \frac{1-x}{2} \right], \quad (31)$$

where ${}_2F_1$ is the classical hypergeometric functions; when $\alpha = \beta = 0$, then the polynomial in (31) becomes the Legendre polynomial (RAINVILLE, 1960, p. 157).

We also have

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!}.$$

Theorem 15. *Integral formula involving Gauss hypergeometric function multiplied with Jacobi polynomials is given by*

$$\begin{aligned} & \int_{-1}^1 x^\lambda (1-x)^\alpha (1+x)^\mu P_n^{(\alpha,\beta)}(x) F_p^{(\delta,\eta)}(a,b;c;kx) dx \\ &= \frac{(-1)^n 2^{\alpha+\mu+1} \beta(\mu+1, n+\alpha+1)}{n! (\mu+\beta+1)_n} \sum_{r=0}^{\infty} \frac{(a)_r \beta_p^{(\delta,\eta)}(b+r, c-b) k^r}{\beta(b, c-b) r!} \\ & \times {}_3F_2 \left[\begin{matrix} -\lambda-r, \mu+\beta+1, \mu+1; \\ \mu+\beta+n+1, \mu+\alpha+n+2; \end{matrix} 1 \right]. \end{aligned} \quad (32)$$

Proof. By using (9), we have

$$\begin{aligned} \mathcal{I}_7 &= \int_{-1}^1 x^\lambda (1-x)^\alpha (1+x)^\mu P_n^{(\alpha,\beta)}(x) \sum_{r=0}^{\infty} (a)_r \frac{\beta_p^{(\delta,\eta)}(b+r, c-b) (kx)^r}{\beta(b, c-b) r!} dx \\ &= \sum_{r=0}^{\infty} \frac{(a)_r \beta_p^{(\delta,\eta)}(b+r, c-b) (k)^r}{\beta(b, c-b) r!} \int_{-1}^1 x^{\lambda+r} (1-x)^\alpha (1+x)^\mu P_n^{(\alpha,\beta)}(x) dx \end{aligned}$$

Next, we use the following formula:

$$\begin{aligned} & \int_{-1}^1 x^\lambda (1-x)^\alpha (1+x)^\mu P_n^{(\alpha,\beta)}(x) dx = (-1)^n \frac{2^{\alpha+\mu+1} \Gamma(\mu+1) \Gamma(n+\alpha+1) \Gamma(\mu+\beta+1)}{n! \Gamma(\mu+\beta+n+1) \Gamma(\mu+\alpha+n+2)} \\ & \times {}_3F_2 \left[\begin{matrix} -\lambda, \mu+\beta+1, \mu+1; \\ \mu+\beta+n+1, \mu+\alpha+n+2; \end{matrix} 1 \right], \end{aligned} \quad (33)$$

where $\alpha > -1$ and $\beta > -1$. Also, ${}_3F_2$ is the special case of generalized hypergeometric series. Then we arrive at

$$\begin{aligned} \mathcal{I}_7 &= \sum_{r=0}^{\infty} \frac{(a)_r \beta_p^{(\delta,\eta)}(b+r, c-b) (k)^r}{\beta(b, c-b) r!} \frac{(-1)^n 2^{\alpha+\mu+1} \Gamma(\mu+1) \Gamma(n+\alpha+1) \Gamma(\mu+\beta+1)}{n! \Gamma(\mu+\beta+n+1) \Gamma(\mu+\alpha+n+2)} \\ & \times {}_3F_2 \left[\begin{matrix} -\lambda-r, \mu+\beta+1, \mu+1; \\ \mu+\beta+n+1, \mu+\alpha+n+2; \end{matrix} 1 \right], \end{aligned}$$

by a little simplification, then we arrive at the desired result in (32). This completes the proof. \square

Integrals involving Gauss hypergeometric function with Legendre function

The Legendre functions are the solution of Legendre’s differential equation (ERDÉLYI et al., 1953, sec.3.1)

$$(1 - z^2) \frac{d^2 f}{dz^2} - 2z \frac{df}{dz} + [\nu(\nu + 1) - \mu^2 (1 - z^2)^{-1}] f = 0, \tag{34}$$

where z, μ, ν are unrestricted.

If we substitute $f = (z^2 - 1)^{\frac{1}{2\mu}} \nu$, then (34) becomes

$$(1 - z^2) \frac{d^2 \nu}{dz^2} - 2(\mu + 1)z \frac{d\nu}{dz} + [\nu(\mu - \nu)(\mu + \nu + 1)] = 0, \tag{35}$$

and with $\delta = \frac{1}{2} - \frac{1}{2}z$ as the independent variable the above differential equation becomes as following:

$$\delta(1 - \delta) \frac{d^2 \nu}{d\delta^2} + (\mu + 1)(1 - 2\delta) \frac{d\nu}{d\delta} + [\nu(\nu - \mu)(\mu + \nu + 1)] = 0. \tag{36}$$

The solution of (34) in the form of Gauss hypergeometric type equation with $a = \mu - \nu, b = \mu + \nu + 1$ and $c = \mu + 1$, as follows.

$$f = P_\nu^\mu(z) = \frac{1}{\Gamma(1 - \mu)} \left(\frac{z + 1}{z - 1}\right)^{\frac{1}{2\mu}} F\left[-\nu, \nu + 1; 1 - \mu; \frac{1}{2} - \frac{1}{2}z\right], \quad |1 - z| < 2, \tag{37}$$

where $P_\nu^\mu(z)$ is known as the Legendre function of the first kind (ERDÉLYI et al., 1953).

Next, we derive the integrals with Legendre function.

Theorem 16. *Integral formula involving Gauss hypergeometric function multiplied with Legendre function is given as following:*

$$\begin{aligned} & \int_0^1 x^{\sigma-1} (1 - x^2)^{\frac{\mu}{2}} P_\nu^\mu(x) F_p^{(\alpha, \beta)}(a, b; c; kx) dx \\ &= \frac{(-1)^\mu 2^{-\sigma-\mu} \sqrt{\pi} \Gamma(1 + \mu + \nu)}{\Gamma(1 - \mu + \nu)} \sum_{n=0}^{\infty} \frac{(a)_n \beta_p^{(\alpha, \beta)}(b + n, c - b) k^n}{\beta(b, c - b) n!} \\ & \times \frac{2^{-n} \Gamma(\sigma + n)}{\Gamma\left(\frac{1}{2} + \frac{\sigma+n}{2} + \frac{\mu}{2} - \frac{\nu}{2}\right) \Gamma\left(1 + \frac{\sigma+n}{2} + \frac{\mu}{2} + \frac{\nu}{2}\right)}. \end{aligned} \tag{38}$$

Proof.

$$\mathcal{I}_8 = \int_0^1 x^{\sigma-1} (1 - x^2)^{\frac{\mu}{2}} P_\nu^\mu(x) \sum_{n=0}^{\infty} \frac{(a)_n \beta_p^{(\alpha, \beta)}(b + n, c - b) (kx)^n}{\beta(b, c - b) n!} dx$$

Next, using the formula (ERDÉLYI et al., 1953, sec. 3.12) for $\Re(\sigma) > 0, \mu \in \mathbb{N}$.

$$\int_0^1 x^{\sigma-1} (1 - x^2)^{\frac{\mu}{2}} P_\nu^\mu(x) dx = \frac{(-1)^\mu 2^{-\sigma-\mu} \sqrt{\pi} \Gamma(\sigma) \Gamma(1 + \mu + \nu)}{\Gamma(1 - \mu + \nu) \Gamma\left(\frac{1}{2} + \frac{\sigma}{2} + \frac{\mu}{2} - \frac{\nu}{2}\right) \Gamma\left(1 + \frac{\sigma}{2} + \frac{\mu}{2} + \frac{\nu}{2}\right)}, \tag{39}$$

then we obtain

$$\mathcal{I}_8 = \sum_{n=0}^{\infty} \frac{(a)_n \beta_p^{(\alpha, \beta)}(b + n, c - b) (k)^n}{\beta(b, c - b) n!} \frac{(-1)^\mu 2^{-\sigma-\mu-n} \sqrt{\pi} \Gamma(\sigma + n) \Gamma(1 + \mu + \nu)}{\Gamma(1 - \mu + \nu) \Gamma\left(\frac{1}{2} + \frac{\sigma+n}{2} + \frac{\mu}{2} - \frac{\nu}{2}\right) \Gamma\left(1 + \frac{\sigma+n}{2} + \frac{\mu}{2} + \frac{\nu}{2}\right)}.$$

This completes the proof. □

Integrals involving Gauss hypergeometric function and Bessel Maitland function

The Bessel Maitland function (also known as Wright generalized Bessel function) defined as following (KIRYAKOVA, 1994):

$$J_{\nu}^{\mu}(z) = \phi(\mu, \nu + 1 : z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\mu n + \nu + 1)} \frac{(-z)^n}{n!}. \quad (40)$$

Theorem 17.

$$\begin{aligned} & \int_0^{\infty} x^{\rho} J_{\nu}^{\mu}(x) F_p^{(\alpha, \beta)}(a, b; c; kx) dx \\ &= \sum_{n=0}^{\infty} \frac{(a)_n \beta_p^{(\alpha, \beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{\Gamma(\rho+n+1)}{\Gamma(1+\nu-\mu-\mu(\rho+n))} \frac{k^n}{n!}. \end{aligned} \quad (41)$$

Proof.

$$I_9 = \sum_{n=0}^{\infty} \frac{(a)_n \beta_p^{(\alpha, \beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{k^n}{n!} \int_0^{\infty} x^{\rho+n} J_{\nu}^{\mu}(x) dx,$$

Next, using the following formula (SAXENA, 2001):

$$\int_0^{\infty} x^{\rho} J_{\nu}^{\mu}(x) dx = \frac{\Gamma(\rho+1)}{\Gamma(1+\nu-\mu-\mu\rho)} \quad (\Re(\rho) > -1, 0 < \mu < 1), \quad (42)$$

then we arrive at the desired result in (41). This completes the proof. \square

Concluding Remarks

We have obtained some new integrals involving Gauss hypergeometric and Confluent hypergeometric function. The results obtained here are basic in nature and are likely to find useful applications in the study of simple and multiple variable hypergeometric series which in turn are useful in statistical mechanics, electrical networks and probability theory. Some important results are also given as special cases of our main results.

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