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A new three-dimensional chaotic system: a single four-scroll attractor

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Abstract: *In this paper, a new three-dimensional autonomous chaotic system is presented with nine terms including three multipliers, which is different from the Lorenz system and other existing systems. Basic dynamical properties of the new system are investigated via equilibria, Lyapunov exponent spectrum, a dissipative system, phase portraits, the Poincaré map and bifurcation diagrams. The compound structures of this new system are also analyzed. The theoretical analysis and numerical simulation validate the main results of this paper.*

Keywords: Chaos; New chaotic system; Lyapunov exponents; Poincaré map; Compound structure.

Introduction

The first research about of three-dimensional chaotic systems was done by Edward N. Lorenz in 1963 (LORENZ, 1963). After that many new three-dimensional chaotic attractors have been proposed by researchers, such as the Rössler system (RÖSSLER, 1976), the Chen system (CHEN; UETA, 1999), the Lü system (LÜ; CHEN, 2002), and the Liu system (LIU et al., 2004). Chaos, as an interesting complex nonlinear phenomenon in nature, can be employed in the vast areas of science and technology, such as information and computer sciences, biomedical systems analysis, flow dynamics and liquid mixing, economics, encryption and secure communications, nonlinear circuits, synchronization, and so on (BUSCARINO et al., 2012; CHEN; DONG, 1998; CHEN; LAI, 1998; GUAN; LIU, 2010; KILIÇ; SARAÇOĞLU; YILDIRIM, 2006; LIU, 2012; LIU; GUAN, 2011; SCHIFF; JERGER; DUONG, 1994; WANG; TANG; ZHANG, 2012; WANG et al., 2002; WU et al., 2007, 2012; ZHANG et al., 2013). Also, in the recent decades fractional order differential equations (see Aminikhah, Refahi Sheikhani and Rezazadeh (2015b, 2016a, 2016b); Mashoof and Refahi Sheikhani (2017a, 2017b), Mashoof, Refahi Sheikhani and Saberi Najafi (2017); Refahi Sheikhani and Mashoof (2017); and Refahi Sheikhani et al. (2012)) especially studying fractional version of these systems is important, for example see Daftardar-Gejji and Bhalekar (2010); and Aminikhah, Refahi Sheikhani and Rezazadeh (2013) and references therein.

In this paper a new three-dimensional chaotic system containing nine terms including three multipliers is proposed to introduce the nonlinearity necessary for folding trajectories. The theoretical analysis and numerical simulation demonstrate

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apparently that proposed system is similar to Lorenz and other chaotic attractors, but its topological structure is different from any existing chaotic attractors.

The rest of this article is organized as follows. Section 2 introduces the design of a new chaotic system. Section 3 describes some fundamental properties of the new system. Section 4 shows forming mechanism of the new chaotic system. Finally, conclusions are given in section 5.

A new three-dimensional chaotic attractor

The new chaotic system is expressed by the following differential equations:

$$\begin{cases} \dot{x} = -ax + byz - cy \\ \dot{y} = y - dxz + kz \\ \dot{z} = -z + mxy - nx \end{cases} \tag{1}$$

where x, y and z are the states and the constants a, b, c, d, k, m , n are positive parameters of the system. The new system (1) has totally nine terms on the right-hand side with three different nonlinear items.

2.1. *Numerical simulation of the new chaotic attractor*

Suppose that $a = 4$, $b = 3$, $c = 0.079$, $d = 7.4$, $k = 0.0808$, $m = 2.011$, $n = 0.08$. With the given parameters and initial conditions $(x_0, y_0, z_0) = (3, 2, 1)$, system (1) has a single 4-scroll chaotic attractor that demonstrates abundant complex behaviors of chaotic dynamics. Using MATLAB program, the numerical simulation has been completed. Figures 1(a)-(d) show the trajectory of the system (1) for a three-dimensional view, an x–y phase plane, an x–z phase plane and a y–z phase plane, respectively. Apparently, the strange attractors in system (1) are different to Lorenz and any existing chaotic attractors. As it is observed this chaotic attractor is a butterfly shaped attractor in x–z phase plane.

Some fundamental properties of the new chaotic system

Equilibria and stability

The equilibria of the new chaotic system (1) can be obtained by solving the equations:

$$\begin{cases} -ax + byz - cy = 0 \\ y - dxz + kz = 0 \\ -z + mxy - nx = 0 \end{cases} \tag{2}$$

The system (2) has five equilibrium points, which are respectively described as follows:

$$O\,(0,0,0)\,, E_1\,(x_1, y_1, z_1)\,, E_2\,(x_2, y_2, z_2)$$

$$E_3\,(x_3, y_3, z_3)\,, E_4\,(x_4, y_4, z_4)$$

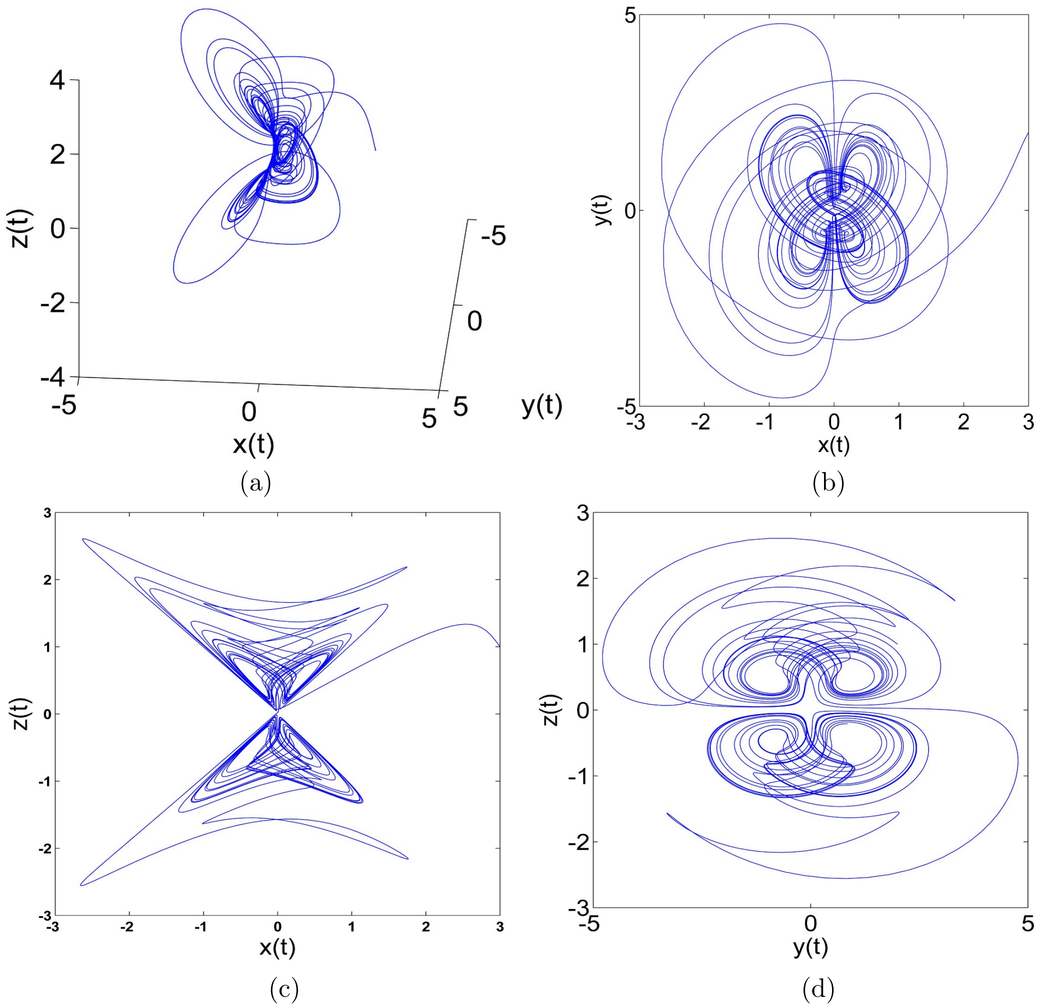


Figure 1: (a) Three-dimensional view. (b) x-y phase plane strange attractors. (c) x-z phase plane strange attractors. (d) y-z phase plane strange attractors.

We operate above these nonlinear algebraic equations and obtain

$$O\,(0,0,0)\,, E_1\,(-0.2484,-0.8230,0.4288)\,, E_2\,(0.2717,0.8619,0.4466)$$

$$E_3\,(-0.2612,0.8113,-0.4029)\,, E_4\,(0.2590,-0.7723,-0.4208)$$

For the first equilibrium $O\,(0,0,0)$, the system (1) is linearized (see for example Khalil (1992); and Slotine and Li (1991)), the Jacobian matrix is defined as

$$J_0 = \begin{bmatrix} -a & bz-c & by \\ -dz & 1 & -dx+k \\ my-n & mx & -1 \end{bmatrix} = \begin{bmatrix} -4 & -0.079 & 0 \\ 0 & 1 & 0.0808 \\ -0.08 & 0 & -1 \end{bmatrix}.$$

By letting $|\lambda I - J_0| = 0$ and solving it (see for example Saberi Najafi; Edalatpanah and Refahi Sheikhani (2014)), the eigenvalues of J_0 are obtained as follows:

$$\lambda_1 = -4, \lambda_2 = -1.0001, \lambda_3 = 1.0001.$$

As it is seen λ_3 is a positive real number, λ_1 and λ_2 are two negative real numbers. Consequently, the equilibrium $O\,(0,0,0)$ is a saddle point and the new system (1) is unstable at O equilibrium point.

Next, linearizing the system (1) about the other equilibria such as E_1, E_2, E_3 and E_4 yields the following characteristic operation.

For equilibrium point E_1, it has a Jacobian matrix equal to

$$J_1 = \begin{bmatrix} -a & bz-c & by \\ -dz & 1 & -dx+k \\ my-n & mx & -1 \end{bmatrix} = \begin{bmatrix} -4 & 1.2075 & -2.4691 \\ -3.1733 & 1 & 1.9193 \\ -1.7351 & -0.4996 & -1 \end{bmatrix}.$$

We let $|\lambda I - J_1| = 0$.

The corresponding eigenvalues of the equilibrium point $E_1(x_1, y_1, z_1)$ are

$$\lambda_1 = -4.7942, \lambda_2 = 0.3971 + 1.7765i, \lambda_3 = 0.3971 - 1.7765i.$$

Here λ_1 is a negative real number; λ_2 and λ_3 become a pair of complex conjugate eigenvalues with positive real parts. The equilibrium point E_1 is a saddle-focus point and this equilibrium point is unstable.

For equilibrium point E_2, we think over corresponding linearization of state equations (1), it has a Jacobian matrix equal to

$$J_2 = \begin{bmatrix} -4 & 1.2609 & 2.5858 \\ -3.3051 & 1 & 1.9298 \\ 1.6533 & 0.5464 & -1 \end{bmatrix}.$$

By letting $|\lambda I - J_2| = 0$, the corresponding eigenvalues of $E_2(x_2, y_2, z_2)$ are

$$\lambda_1 = -4.7729, \lambda_2 = 0.3864 + 1.8672i, \lambda_3 = 0.3864 - 1.8672i.$$

Results show that λ_1 is a negative real number, λ_2 and λ_3 form a complex conjugate pair and their real parts are positive. Equilibrium point of E_2 is also a saddle-focus point and this equilibrium point is unstable.

For equilibrium point E_3, has a Jacobian matrix equal to

$$J_3 = \begin{bmatrix} -4 & -1.2877 & 2.4339 \\ 2.9815 & 1 & 2.0136 \\ 1.5515 & -0.5253 & -1 \end{bmatrix}.$$

We let $|\lambda I - J_3| = 0$, the corresponding eigenvalues of the equilibrium point $E_3(x_3, y_3, z_3)$ are

$$\lambda_1 = -4.6878, \lambda_2 = 0.3439 + 1.7963i, \lambda_3 = 0.3439 - 1.7963i.$$

Here λ_1 is a negative real number, λ_2 and λ_3 form a complex conjugate pair and their real parts are positive. Equilibrium point of E_3 is also a saddle-focus point and this equilibrium point is unstable.

For equilibrium point E_4, has a Jacobian matrix equal to

$$J_4 = \begin{bmatrix} -4 & -1.3412 & -2.3170 \\ 3.1136 & 1 & -1.8357 \\ -1.6332 & 0.5208 & -1 \end{bmatrix}.$$

By letting $|\lambda I - J_4| = 0$, the corresponding eigenvalues of $E_4(x_4, y_4, z_4)$ are

$$\lambda_1 = -4.6461, \lambda_2 = 0.3230 + 1.8015i, \lambda_3 = 0.3230 - 1.8015i.$$

Results show that λ_1 is a negative real number, λ_2 and λ_3 form a complex conjugate pair and their real parts are positive. Equilibrium point of E_4 is also a saddle-focus point and so, this equilibrium point is unstable.

The observations show that the five equilibrium points of the nonlinear systems are all saddle focus-nodes.

A dissipative system and the existence of attractor

The divergence of flow of the system (1) is given by

$$div\ V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -a + 1 - 1 = -a = p$$

where $p = -4$. Since p is less than zero, therefore the system (1) is a dissipative with an exponential rate of contraction as

$$\frac{dV}{dt} = e^{pt} = e^{-4t}.$$

As a result

$$V(t) = V(0)\, e^{pt} = V(0)\, e^{-4t}.$$

This means that any initial volume V_0 containing the system trajectories becomes zero as $t \to \infty$ at an exponential rate of -4. Therefore, all this dynamical system orbits are ultimately confined to a specific limit set of zero-volume, and the asymptotic motion settles onto an attractor in the three-dimensional phase plane of the system (1). So, we conclude the dynamical system finally goes toward an attractor as $t \to \infty$.

Lyapunov exponents and Lyapunov dimension

Chaotic systems are described by how rapidly close initial conditions diverge from each other. One way to calculate this rate of separation is via the Lyapunov exponents. The Lyapunov exponents measure the average exponential rates of divergence or convergence of nearby trajectories in phase space. A system with at least one positive Lyapunov exponent is said to be chaotic (STROGATZ, 1994).

When $a = 4$, $b = 3$, $c = 0.079$, $d = 7.4$, $k = 0.0808$, $m = 2.011$, $n = 0.08$ by applying the Wolf algorithm (WOLF et al., 1985), the Lyapunov exponents of the new system (1) are determined with a normalized step-sized $h = 0.01$ as follows (see figure 2):

$$\lambda_{L1} = 0.2736, \lambda_{L2} = -0.01318, \lambda_{L3} = -4.260.$$

Since one of the Lyapunov exponents is positive, thus the proposed nonlinear system is chaotic. In addition, the Lyapunov dimension of this attractor is

$$D_L = j + \frac{1}{|\lambda_{Lj+1}|}\sum_{i=1}^{j} \lambda_{Li} = 2 + \frac{(\lambda_{L1}+\lambda_{L2})}{|\lambda_{L3}|} = 2 + \frac{0.2736 + (-0.01318)}{|-4.260|} = 2.0611.$$

As it is observed, the Lyapunov dimension of the system (1) is fractional.

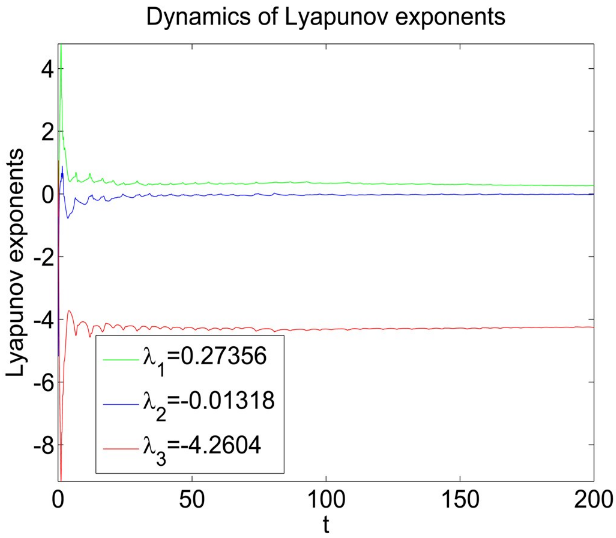


Figure 2: The Lyapunov exponents of the new chaotic system.

Poincaré maps, waveforms, spectrum and bifurcation diagram

The dynamical behaviors of the new chaotic attractor can be further demonstrated by means of Poincaré maps, waveforms, spectrum and bifurcation diagram. Figures 3(a)-(c) show the Poincaré maps in the planes where $x = 0$, $y = 0$ and $z = 0$ respectively. It is seen that the Poincaré maps of the attractors in figures 3(a) and 3(b) consist of symmetrical and almost symmetrical branches. In figure 3(c) the Poincaré map demonstrates a diagonal distribution. The reason for it is found from the fact that the waveforms of $x(t)$ and $y(t)$ have almost the same behavior. Figure 4 illustrates the waveform of $y(t)$ in the time domain. The waveform of $y(t)$

is not periodic. The power spectrums of the signal $y(t)$ of the system (1) and Lorenz system (LORENZ, 1693) are given in figures 5(a) and 5(b).

Figure 6 shows a bifurcation diagram of the system (1) derived by studying the peak of y ('y max') when parameter a varies from 1.8 to 6 with $b = 3$, $c = 0.079$, $d = 7.4$, $k = 0.0808$, $m = 2.011$, $n = 0.08$. In this bifurcation diagram we see some periodic windows in the chaotic region.

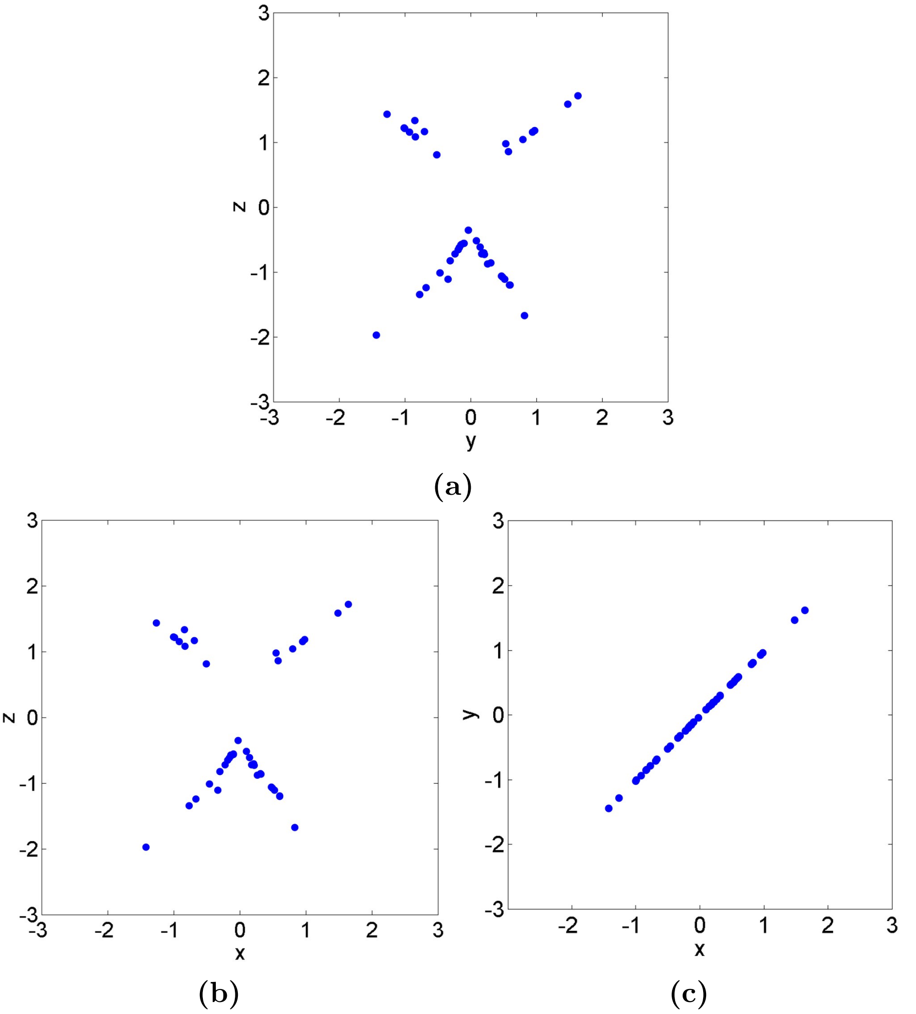


Figure 3: Poincaré maps in the planes where (a) $x = 0$, (b) $y = 0$, (c) $z = 0$.

Forming mechanism of new chaotic system

In order to investigate the compound structures (LÜ; CHEN; ZHANG, 2002) of the new system (1), let us consider its controlled system as below:

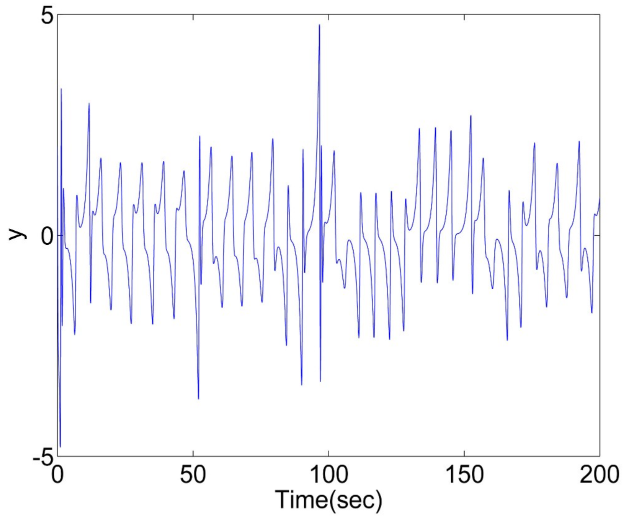


Figure 4: A chaotic waveform of $y(t)$ in the time domain

$$\begin{cases} \dot{x} = -ax + byz - cy \\ \dot{y} = y - dxz + kz + u \\ \dot{z} = -z + mxy - nx \end{cases} \tag{3}$$

In system (3) by changing the parameter u, considered as the "controller", one can see different dynamical behaviors. It should be mentioned that the value of u can be changed within a certain range.

In here, we will consider the parameters and initial values of the system (3) as before.

When $u = 0.47$, the corresponding attractor of the controlled system is evolved into the single right up scroll attractor; it is only one quarter-image of the original attractor in this time (see figure 7(a)). Now, assume that $u = -0.28$, the corresponding attractor of the controlled system is evolved into the single right down scroll attractor; it is only one quarter-image of the original attractor in this time (see figure 7(b)).

A summary of the parameter range for dynamical behaviors of the controlled system (3) is as follows:

- When $u \geq 4.63$, the system (3) converges to a point. Figure 8(a) illustrates convergence to a point at $u = 4.63$.
- When $0.77 \leq u \leq 4.62$, the system (3) has limit cycles. Figure 8(b) illustrates a limit cycles at $u = 2.69$.
- When $0.53 \leq u \leq 0.76$, the system (3) evolves into period-doubling bifurcations. Figure 8(c) illustrates a period-doubling bifurcation at $u = 0.75$.

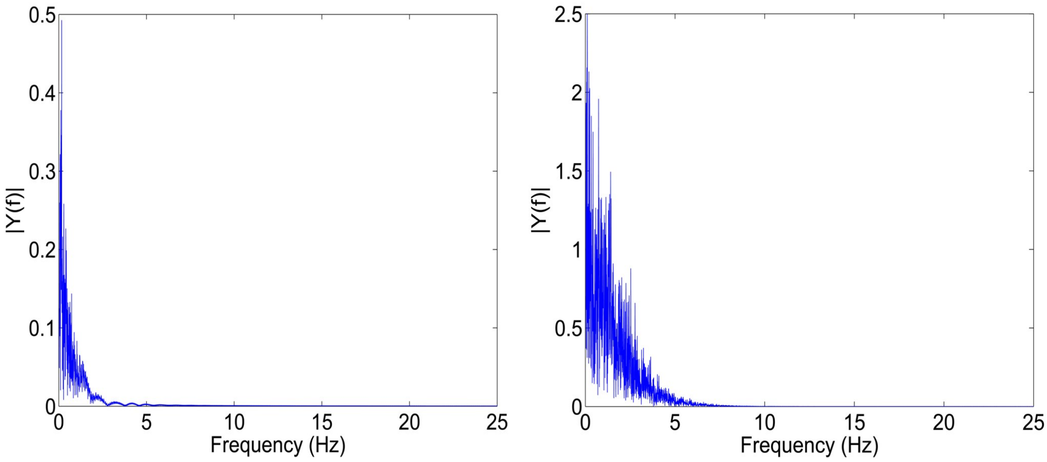


Figure 5: Power spectrum of the signal $y(t)$ of (a) the new system (1), (b) Lorenz system.

- When $0.41 \leq u \leq 0.52$, the system (3) becomes a right up (or a right down) quarter-image attractor. For example, see Figures 7(a) ($u = 0.47$) and 7(b) ($u = -0.28$).
- When $0.01 \leq u \leq 0.40$, the system (3) displays partial attractors, which are bounded in this time. Figure 8(d) illustrates a partial attractor at $u = 0.15$.
- When $u = 0$, the system (3) demonstrates a complete attractor. Figure 1 illustrates a complete attractor.

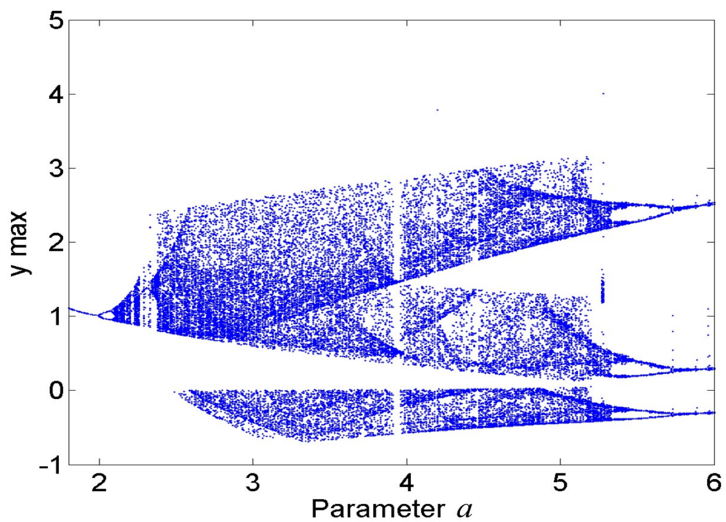
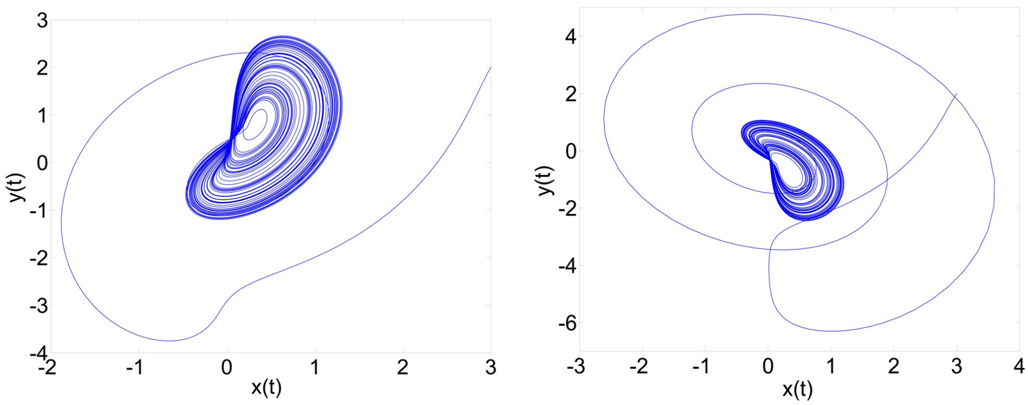


Figure 6: A bifurcation diagram of the new system (1) obtained by studying the peak of y ('y max') versus the parameter a.

Conclusion

The current article proposed a new three-dimensional autonomous chaotic system. The new attractor consists of nine terms in three first-order autonomous ordinary differential equations with three multipliers. The new system was analyzed both theoretically and numerically by studying some fundamental dynamical characteristics such as equilibria, Lyapunov exponent spectrum, a dissipative system, phase portraits, the Poincaré map, bifurcation diagrams and compound structures. The results demonstrated the new attractor is different from the Lorenz attractor and any existing chaotic attractors.

During recent decades theory of fractional order derivatives gained a special interest (see Aminikhah, Refahi Sheikhani and Rezazadeh (2015a); Ansari and Refahi Sheikhani (2014); and Rezazadeh, Aminikhah and Refahi Sheikhani (2016)). As future work, the system reported in this article can be also investigated from this perspective.



(a) **(b)**

Figure 7: x-y phase plane of the system (3) at (a) $u = 0.47$, (b) $u = -0.28$.

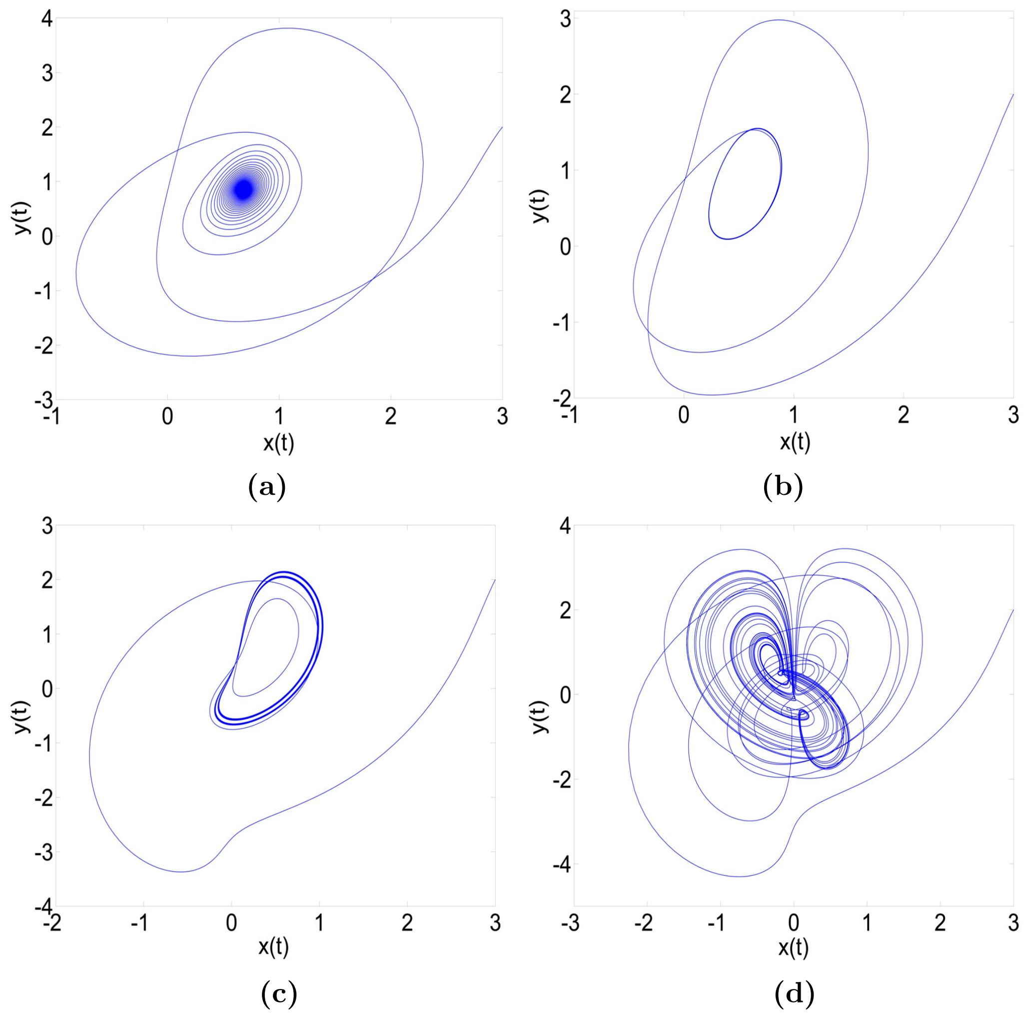


Figure 8: x-y phase plane of the system (3) at (a) $u = 4.63$, (b) $u = 2.69$, (c) $u = 0.75$, (d) $u = 0.15$.

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