A Trigonometry Approach to Balancing Numbers and Their Related Sequences

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Abstract

The balancing numbers satisfy the second order linear homogeneous difference equation $B_{n+1} = 6B_n - B_{n-1}$, on the other hand the Fibonacci numbers are solution of the second order linear homogeneous difference equation $F_{n+1} = F_n + F_{n-1}$, where B_n and F_n denote the n^{th} balancing number and n^{th} Fibonacci number respectively. In a paper, Smith introduce Fibonometry in connection with a differential equation called Fibonometric differential equation. In this study, we first introduce the balancometric differential equation and then obtain the balancometric functions as solutions of this equation.

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1 Introduction

The study of number sequences has been a source of attraction to the mathematicians since ancient times. Since then many of them are focusing their interest on the study of the fascinating triangular numbers. Behera and Panda [1] defined balancing numbers n as solutions of the Diophantine equation $1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r)$, calling r as balancer corresponding to n. By slightly modifying the Diophantine equation, Panda and Ray [5] introduce cobalancing numbers and cobalancers as solution of the Diophantine equation $1+2+\ldots+n = (n+1)+(n+2)+\ldots+(n+R)$, calling$R \in \mathbb{Z}^+$ as the cobalancer corresponding to n. The cobalancing numbers are linked to a third category of triangular numbers that are expressible as product of two consecutive natural numbers (approximately as the arithmetic mean of squares of two consecutive natural numbers i.e. $\frac{n^2+(n+1)^2}{2} \approx n(n+1)$.

In a subsequent paper, Liptai [2] added another interesting result to the theory of balancing numbers by proving that the only balancing number in the Fibonacci sequence is 1. Also in [3], he proved that there are no Lucas balancing numbers in the sequence of Fibonacci numbers. Subsequently, Panda [6] established many fascinating properties of balancing numbers, where he has proved some results resembling trigonometric identities. Many interesting results of balancing numbers and its related sequences are available in the literature [4–9, 12**?** , 13].

Motivated by Fibonometry introduced by Smith [11], in this paper, the concept of balancometry is introduced. The balancometric functions are being obtained from a second order linear differential equation $y'' - 6y' + y = 0$, which we call as balancometric differential equation. Clearly, the differential equations

$$y'' + y = 0,\ y(0) = 0,\ y'(0) = 1 \ \text{ and } \ y'' - y = 0,\ y(0) = 0,\ y'(0) = 1$$

lead to the circular trigonometry and the hyperbolic trigonometry respectively. On the other hand, the initial value problem

$$y'' - 6y' + y = 0,\ y(0) = 0,\ y'(0) = 1 \tag{1.1}$$

known as balancometric differential equation is analogue to the well known recursion formula for balancing numbers

$$B_n = 6B_{n-1} - B_{n-2},\ B_0 = 0,\ B_1 = 1,\ n \geq 2. \tag{1.2}$$

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It is well known that the sine and cosine functions are solution of the differential equation $y^{''}+y=0$. This paper investigates some of the topics associated with the trigonometry derived from the balancometric differential equation. The functions to be developed will be defined as the balancometric functions.

2 Balancometric sine and cosine functions

The solution of balancometric differential equation is $y=\frac{e^{\lambda_1 x}-e^{\lambda_2 x}}{\lambda_1-\lambda_2}$ where λ_1 and λ_2 are indeed, the solutions of the equation $\lambda^2-6\lambda+1=0$. Based on the well known pattern for the classical circular and hyperbolic trigonometry functions

$$\sin x=\frac{e^{ix}-e^{-ix}}{2i},\quad \cos x=\frac{e^{ix}+e^{-ix}}{2i},\quad \sinh x=\frac{e^{x}-e^{-x}}{2} \text{ and } \cosh x=\frac{e^{x}+e^{-x}}{2},$$

the balancometric sine and balancometric cosine are defined as follows:

Definition 2.1. The balancometric sine function is

$$\sin Bx=\frac{e^{\lambda_1 x}-e^{\lambda_2 x}}{\lambda_1-\lambda_2},$$

where $\lambda_1=3+\sqrt{8}$ and $\lambda_2=3-\sqrt{8}$.

It can be observed that, if $\sum\limits_{n=0}^{\infty}a_n x^n$ is a power series solution of the balancing differential equation (1.1), the recursion relation will be

$$a_{n+2}=\frac{6(n+1)a_{n+1}-a_n}{(n+1)(n+2)}. \tag{2.1}$$

With the help of the recursion formula (2.1), we develop an interesting relation between the power series coefficients and the sequence of balancing numbers as follows.

Theorem 2.2. *For n^{th} balancing number B_n,*

$$a_n=\frac{B_n a_1-B_{n-1}a_0}{n!}.$$

Proof. Mathematical induction comes into the picture to prove this theorem. Clearly, the theorem is true for $n=1$ as $a_2=\frac{B_2a_1-B_1a_0}{2!}=\frac{6a_1-a_0}{2!}$ by (2.1). Assume that it holds for all $n\leq(k+1)$ where $k\in\mathbb{Z}^+$. Now, by (2.1) and from the hypothesis, we obtain

$$\begin{aligned}(k+1)(k+2)a_{k+2}&=6(k+1)a_{k+1}-a_k\\&=6\left[\frac{B_{k+1}a_1-B_ka_0}{k!}\right]-\frac{B_ka_1-B_{k-1}a_0}{k!}\\&=\frac{(6B_{k+1}-B_k)a_1-(6B_k-B_{k-1})a_0}{k!}=\frac{B_{k+2}a_1-B_{k+1}a_0}{k!},\end{aligned}$$

which follows the proof of the theorem. □

Following theorem establishes an alternating expression for balancometric sine function.

Theorem 2.3. *The balancometric sine function is*

$$\sin Bx=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!},$$

where B_n is the n^{th} balancing number.

Proof. By virtue of Theorem 2.2, the solution of the differential equation (1.1) will take the form

$$y = a_0 + a_1 x + \left(\frac{6a_1 - a_0}{2!}\right) x^2 + \ldots + \left(\frac{B_n a_1 - B_{n-1} a_0}{n!}\right) x^n + \ldots.$$

Applying the initial conditions, we get $a_0 = 0$ and $a_1 = 1$ and therefore, we obtain

$$\begin{aligned} y &= x + \left(\frac{6}{2!}\right) x^2 + \left(\frac{35}{3!}\right) x^3 + \ldots + \left(\frac{B_n a_1 - B_{n-1} a_0}{n!}\right) x^n + \ldots \\ &= \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \end{aligned}$$

which completes the proof. □

Following theorem establishes the convergence of the balancometric sine function.

Theorem 2.4. *The series expansion for* $\sin Bx$ *is absolutely convergent for all* $x \in \mathbb{R}$.

Proof. By ratio test for convergence and since $\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = \lambda_1 = 3 + \sqrt{8}$, we obtain

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{\frac{B_{n+1}}{(n+1)!}}{\frac{B_n}{n!}} \right| &= \lim_{n\to\infty} \left[\frac{1}{(n+1)} \frac{B_{n+1}}{B_n} \right] \\ &= 0 \cdot \lim_{n\to\infty} \frac{B_{n+1}}{B_n} = 0 \cdot \lambda_1 = 0, \end{aligned}$$

implies the series $|\sin Bx|$ is convergent. Thus the series expansion for $\sin Bx$ is absolutely convergent for all $x \in \mathbb{R}$. □

Now, we introduce the balancometric cosine function as the derivative of balancometric sine function, that is, $\cos Bx = \frac{d}{dx} \sin Bx$, or equivalently, from Theorem 2.3, we have the following definition.

Definition 2.5. The balancometric cosine function is

$$\cos Bx = \frac{\lambda_1 e^{\lambda_1 x} - \lambda_2 e^{\lambda_2 x}}{\lambda_1 - \lambda_2} = \sum_{n=0}^{\infty} B_{n+1} \frac{x^n}{n!}$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.

Since the derivative of an absolutely convergent series is convergent, the balancometric cosine function $\cos Bx$ is also convergent.

3 Balancometric tangent and cotangent

In this section, we will follow the familiar patterns for the circular and hyperbolic functions and define the analogous balancometric functions such as balancometric tangent and cotangent functions. Similar to the definitions of classical tangent and cotangent functions, we introduce balancometric tangent and balancometric cotangent functions as follows.

Definition 3.1. The balancometric tangent and balancometric cotangent functions are defined respectively by

$$\tan Bx = \frac{\sin Bx}{\cos Bx} \quad \text{and} \quad \cot Bx = \frac{\cos Bx}{\sin Bx}, \quad \sin Bx \neq 0.$$

Since λ_1 and λ_2 are roots of the equation $\lambda^2 - 6\lambda + 1 = 0$, we notice that, $\lambda_1^2 = 6\lambda_1 - 1$ and using this, we get $\lambda_1^3 = \lambda_1(6\lambda_1 - 1) = 6(6\lambda_1 - 1) - \lambda_1 = 35\lambda_1 - 6$. Thus, we suggest the following proposition.

Proposition 3.2. *For $n \geq 1$,*

$$(i)\ \lambda_1^n = B_n\lambda_1 - B_{n-1},$$
$$(ii)\ \lambda_2^n = B_n\lambda_2 - B_{n-1}.$$

Proof. Once again induction play the role to prove this proposition. Clearly, it holds for $n = 1, 2$. Assume that it is true for all $n \leq k$. Now by the recurrence relation (1.2) and from the hypothesis, we obtain

$$\begin{aligned}\lambda_1^{k+1} &= \lambda_1(B_k\lambda_1 - B_{k-1})\\ &= B_k(6\lambda_1 - 1) - \lambda_1 B_{k-1}\\ &= B_{k+1}\lambda_1 - B_k.\end{aligned}$$

This completes the proof of (i). Similarly one can prove (ii). □

The following theorems demonstrate the alternative expressions for balancometric tangent and cotangent functions.

Theorem 3.3. *The balancometric tangent function is of the form*

$$\tan Bx = \lambda_2 + (34\lambda_2 - 6)\left[\sum_{n=0}^{\infty} (B_n\lambda_2 - B_{n-1})^2 e^{-2\sqrt{8}nx}\right],$$

where $\lambda_2 = 3 - \sqrt{8}$.

Proof. By virtue of Definition 3.1, we have

$$\tan Bx = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 e^{\lambda_1 x} - \lambda_2 e^{\lambda_2 x}} = \frac{1 - e^{(\lambda_2 - \lambda_1)x}}{\lambda_1 - \lambda_2 e^{(\lambda_2-\lambda_1)x}}.$$

Dividing λ_1 into the numerator and denominator and using the fact $\lambda_1\lambda_2 = 1,\ \lambda_1 - \lambda_2 = 2\sqrt{8}$, we obtain

$$\tan Bx = \lambda_2(1 - e^{-2\sqrt{8}x})\left(1 - \lambda_2^2 e^{-2\sqrt{8}x}\right)^{-1}.$$

Expanding the inverse expression and replacing $\lambda_2^2 - 1$ with $6\lambda_2 - 2$ (as λ_2 is a solution of $\lambda^2 - 6\lambda + 1 = 0$), we get

$$\begin{aligned}\tan Bx &= \lambda_2\left[1 + (6\lambda_2 - 2)e^{-2\sqrt{8}x} + \lambda_2^2(6\lambda_2 - 2)e^{-4\sqrt{8}x} + \lambda_2^4(6\lambda_2 - 2)e^{-6\sqrt{8}x} + \dots\right]\\ &= \lambda_2 + (6\lambda_2^2 - 2\lambda_2)\sum_{n=0}^{\infty} \lambda_2^{2n} e^{-2\sqrt{8}nx}\\ &= \lambda_2 + (34\lambda_2 - 6)\sum_{n=0}^{\infty} \lambda_2^{2n} e^{-2\sqrt{8}nx},\end{aligned}$$

and therefore the theorem follows from Proposition 3.2. □

We can get a similar expression for $\tan Bx$ in terms of λ_1 also.

As a bonus, we observe that the Binet's formula for balancing number can also be derived from Proposition 3.2 by subtracting (ii) from (i).

Theorem 3.4. *The balancometric cotangent function is of the form*

$$\cot Bx = \frac{1}{\lambda_2}\left[1 - (6\lambda_2 - 2)\sum_{n=0}^{\infty} e^{-2\sqrt{8}nx}\right],$$

where $\lambda_2 = 3 - \sqrt{8}$.

Proof. Again by Definition 3.1, we have

$$\cot Bx = \frac{\lambda_1 e^{\lambda_1 x} - \lambda_2 e^{\lambda_2 x}}{e^{\lambda_1 x} - e^{\lambda_2 x}}.$$

Dividing $\lambda_1 e^{\lambda_1 x}$ into the numerator and denominator and using the fact $\lambda_1\lambda_2 = 1,\ \lambda_1 - \lambda_2 = 2\sqrt{8}$, we obtain

$$\cot Bx = \frac{1 - \lambda_2^2 e^{-2\sqrt{8}x}}{\lambda_2 - \lambda_2 e^{-2\sqrt{8}x}} = \frac{1}{\lambda_2}\left[1 - (6\lambda_2 - 1)\left(1 - e^{-2\sqrt{8}x}\right)^{-1}\right].$$

Expanding the inverse expression, we get

$$\begin{aligned}\cot Bx &= \frac{1}{\lambda_2}\left[1 - (6\lambda_2 - 2)e^{-2\sqrt{8}x} - (6\lambda_2 - 2)e^{-4\sqrt{8}x} - \dots\right]\\ &= \frac{1}{\lambda_2}\left[1 - (6\lambda_2 - 2)\sum_{n=0}^{\infty} e^{-2\sqrt{8}nx}\right],\end{aligned}$$

which ends the proof. □

Remark 3.5. For $x = 0$, the series $\cot Bx$ is undefined as $\sin Bx = 0$. That is, the series fails to converge since $(6\lambda_2 - 2)\sum_{n=0}^{\infty} 1 = (6\lambda_2 - 2)\lim_{n\to\infty} n = \infty$. Further, for any $x < 0$, each of the terms in the series for $\cot Bx$ is a positive power of e, and hence, is greater than 1. Thus, the series diverges for$x \leq 1$. More precisely, the series $\cot Bx$ converges for $x > 0$.

Definitions for the balancing secant and cosecant can be obtained analogously and their series are investigated.

4 Fundamental identities of balancometric functions

In circular trigonometry, the identity $\sin^2 x + \cos^2 x = 1$ leads to the circle $x^2 + y^2 = 1$ with $x = \sin x,\ y = \cos x$ whereas in hyperbolic trigonometry, the identity $\cosh^2 x - \sinh^2 x = 1$ leads to the hyperbola $x^2 - y^2 = 1$ with $x = \cosh x,\ y = \sinh x$. However, the situation is not so direct in balancometric functions. The following theorem demonstrates this fact.

Theorem 4.1. *The fundamental identities for balancometric functions is*

$$\cos B^2 x - 6\cos Bx \sin Bx + \sin B^2 x = e^{6x}.$$

Proof. By virtue of definitions, Definition 2.1 and Definition 2.5 and since $\lambda^2 - 6\lambda + 1 = 0$ for $\lambda_1 = 3 + \sqrt{8},\ \lambda_2 = 3 - \sqrt{8}$, we have

$$\begin{aligned}\cos B^2 x - 6\cos Bx \sin Bx + \sin B^2 x &= \left[\frac{\lambda_1 e^{\lambda_1 x} - \lambda_2 e^{\lambda_2 x}}{\lambda_1 - \lambda_2}\right]^2 - 6\frac{\lambda_1 e^{\lambda_1 x} - \lambda_2 e^{\lambda_2 x}}{\lambda_1 - \lambda_2}\frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2} + \left[\frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2}\right]^2\\ &= \frac{(\lambda_1^2 - 6\lambda_1 + 1)e^{2\lambda_1 x}}{(\lambda_1 - \lambda_2)^2} + \frac{(-2 + 6(\lambda_1 + \lambda_2) - 2)e^{(\lambda_1 + \lambda_2)x}}{(\lambda_1 - \lambda_2)^2} + \frac{(\lambda_2^2 - 6\lambda_2 + 1)e^{2\lambda_2 x}}{(\lambda_1 - \lambda_2)^2}\\ &= e^{6x},\end{aligned}$$

which completes the proof. □

Remark 4.2. The cobalancing numbers, Lucas-balancing numbers and Lucas-cobalancing numbers are defined recursively by $b_{n+1} = 6b_n - b_{n-1} + 2$, $C_{n+1} = 6C_n - C_{n-1}$ and $c_{n+1} = 6c_n - c_{n-1}$ respectively subject to the respective initial conditions $b_0 = 0,\ b_1 = 2,\ C_0 = 1,\ C_1 = 3$ and $c_0 = 1,\ c_1 = 7$. Arguing as above, we may deduce the cobalanciometric sine, Lucas-balancometric sine and Lucas-cobalancometric sine respectively as

$$\sin bx = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!},\ \text{where } d_n = b_n + \frac{1}{2},\ \ \sin LBx = \sum_{n=0}^{\infty} C_n \frac{x^n}{n!}\ \text{ and }\ \sin Lbx = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}.$$

Similarly the cobalancometric cosine, Lucas-balancometric cosine and Lucas-cobalancometric cosine respectively as

$$\cos bx = \sum_{n=0}^{\infty} d_{n+1}\frac{x^n}{n!}, \text{ where } d_{n+1} = b_{n+1} + \frac{1}{2}, \ \cos LBx = \sum_{n=0}^{\infty} C_{n+1}\frac{x^n}{n!} \text{ and } \cos Lbx = \sum_{n=0}^{\infty} c_{n+1}\frac{x^n}{n!}.$$

References

[1] Behera, A.; Panda, G.K,, On the square roots of triangular numbers, *The Fibonacci Quarterly*, 37(2), 1999, 98-105.

[2] Liptai, K., Fibonacci balancing numbers, *The Fibonacci Quarterly*, 42(4), 2004, 330-340.

[3] Liptai, K., Lucas balancing numbers, *Acta Math.Univ. Ostrav*, 14(1), 2006, 43-47.

[4] Panda, G.K.; Ray, P.K., Some links of balancing and cobalancing numbers with Pell and associated Pell numbers, *Bulletin of the Institute of Mathematics, Academia Sinica (New Series)*, 6(1), 2011, 41-72.

[5] Panda, G.K.; Ray, P.K., Cobalancing numbers and cobalancers, *International Journal of Mathematics and Mathematical Sciences*, 2005(8), 2005, 1189-1200.

[6] Panda, G.K. Some fascinating properties of balancing numbers, *Proc. Eleventh Internat. Conference on Fibonacci Numbers and Their Applications, Cong. Numerantium*, 194, 2009, 185-189.

[7] Ray, P.K., Application of Chybeshev polynomials in factorization of balancing and Lucas-balancing numbers, *Bol. Soc. Paran. Mat.* Vol.30 (2), 2012, 49-56.

[8] Ray, P.K., Factorization of negatively subscripted balancing and Lucas-balancing numbers, *Bol.Soc.Paran.Mat.*, Vol.31 (2), 2013, 161-173.

[9] Ray, P.K., Curious congruences for balancing numbers, *Int.J.Contemp.Sciences*, Vol.7 (18), 2012, 881-889.

[10] Ray, P.K., Certain matrices associated with balancing and Lucas-balancing numbers, *Matematika*, Vol.28 (1), 2012, 15-22.

[11] Ray, P.K., New identities for the common factors of balancing and Lucas-balancing numbers, International Journal of Pure and Applied Mathematics, Vol. 85, 487-494 (2013).

[12] Ray, P.K., Some congruences for balancing and Lucas-balancing numbers and their applications, Integers, 14, 2014, #A8.

[13] Ray, P.K., On the properties of Lucas-balancing numbers by matrix method, Sigmae, Alfenas, 3(1), 2014, 1-6.

[14] Smith, R.M., Introduction to analytic Fibonometry, *Alabama Journal of Mathematics*, 2001, 27-36.