ISSN: 2317-0840

Modelling and Simulations of a Couple Stress Fluid Flow and Heat Transfer in Inclined Geometry

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Abstract: *The study of couple stress fluid flow and heat transfer in inclined geometries is crucial in understanding various engineering applications, such as heat exchangers, lubrication systems, and biomedical devices. The interplay of fluid flow, heat transfer, and geometry creates complex, challenging problems that make accurate prediction of these phenomena difficult, requiring advanced techniques and models. This study aims to address these challenges by developing a comprehensive numerical model for simulating couple stress fluid flow and heat transfer in inclined geometry. From the momentum, continuity, and energy equations, we have derived strongly nonlinear ordinary differential equations. The Homotopy Perturbation Method (HPM) is applied to solve the governing coupled nonlinear differential equations with appropriate boundary conditions. The solutions yield expressions for the velocity profile, temperature distribution, vorticity, volume flow rate, shear stress, and average velocity. Numerical and graphical comparisons of the effects of various parameters on temperature and velocity show good agreement with exact solutions.*

Keywords: *Couple stress fluid; Thin film flow; Homotopy perturbation method; Heat transfer.*

Introduction

Non-Newtonian fluids are examined from both theoretical and applied perspectives has piqued the interest of people worldwide in recent years (Fetecau and Fetecau 2003, Ai and Vafai 2005, Islam and Zhou 2007, Alam, Siddiqui et al. 2012, Shah, Islam et al. 2012, Farooq, Rahim et al. 2013). Because non-Newtonian fluids are used in many industrial and technical processes, studying them is crucial. Shade, shampoo, dirt, ketchup, polymer melts, blood, clay coatings, certain oils and greases, and several mixtures are well-known examples of these fluids. Such fluids require careful flow analysis, both theoretically and practically. These kinds of flows are theoretically crucial to fluid mechanics. Researchers investigate non-Newtonian fluids in great detail, mostly through the analysis of the resulting differential equations. Fluid mechanics is approached in applied fields like physics and atmospheric rheology, where an experimental setup results in the measurement of material coefficients. The wide range of physical structures seen in non-Newtonian fluids makes it difficult to provide a single constitutive equation that includes all of their characteristics. As a result, several fluid models have been put out to forecast the non-Newtonian behavior of various

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Manuscript received: 12/02/2025; Revised: 08/04/2025; Accepted: 09/04/2025.

material kinds. Among them, the generalized second-grade fluid model has received much interest (Mahesh, Kammappa et al. 2025, Verma 2025).

Newly, a number of researchers have become interested in thin film flows. This is as a result of its wide acceptance in industrial manufacturing processes. However, despite the fact that here is a wealth of works on thin film flows for Newtonian fluids, non-Newtonian fluids have received little attention in this regard (Munson, Young et al. 2006). Rarely have (Siddiqui, Mahmood et al. 2006, Siddiqui, Mahmood et al. 2006, Siddiqui, Ahmed et al. 2007, Siddiqui, Mahmood et al. 2008) and (Hayat and Sajid 2007, Sajid, Ali et al. 2009) attempted to deal with non-Newtonian fluid thin film flows.

The flow and heat transfer within thin films have drawn the attention of numerous investigators (Weinstein, Ruschak et al. 2003, Gul and Khan 2023). This is because of the wide range of technical and industrial uses they have, including the processing of food products, coatings for fiber and wire, fluidization of reactors, transpiration cooling, processing of polymers, heat pipes, gaseous diffusion and fluidic cells found in numerous biochemical and organic finding systems. The matter of spaces for natural and living discovery systems, like as fluidic cells for organic and biological micro cantilevers, was examined by (Ullah, Alam et al. 2024). The power-law fluid model is used as the non-Newtonian fluid in the majority of flow and heat transfer study problems. Studies that combine the impacts of viscous dissipation have received only modest attention, despite the fact that numerous instances, such as polymer processing, have demonstrated the significance of this phenomenon.

The primary objective of this study is to develop a comprehensive numerical model for simulating couple stress fluid flow and heat transfer in inclined geometry with a specific focus on understanding the effects fluid properties on heat transfer rates and fluid flow patterns. This paper is organized as follows. The basic equations are discussed in section 2. The problem formulation is given in section 3. Section 4 discusses the basic phenomena of the method (HPM) and the problem's solution. Volumetric flow rate, shear stress, vorticity, and average velocity of the problem are given in section 5, while the results and discussion are addressed in section 6 and section 7 provides the conclusion of the paper.

Basic equations

The main idea equations guiding a couple stress fluids' flow, taking into consideration thermal effects (Eldabe, Hassan et al. 2003, Siddiqui, Ahmed et al. 2006, Islam and Zhou 2007, Siddiqui, Zeb et al. 2008, Islam, Ali et al. 2009) are

$$\nabla . u = 0, \tag{1}$$

$$\rho \frac{Du}{Dt} = \rho f - \nabla P - \nabla . T - \eta \nabla^4 u, \tag{2}$$

$$\rho C_p \frac{D\Psi}{Dt} = \kappa \nabla^2 \Psi - T . L; \tag{3}$$

The fluid velocity is symbolizes by u , the body force is signified by f, the couple stress parameter is denotedη, the fluid density is represents byρ , "P " is the pressure, Ψ stand for temperature, the specific heat is signified by, \\\\\\{C\\\\\\} rsub \\\\\\{p\\\\\\} , the material time derivative denoted \\\\\\{D \\\\\\} over \\\\\\{Dt\\\\\\} and κ stands for conductivity. These values are clear as

$$\frac{D}{Dt}=\left(\frac{\partial}{\partial t}+u\,.\,\nabla\right),$$

where

$$T=A_1\mu,$$

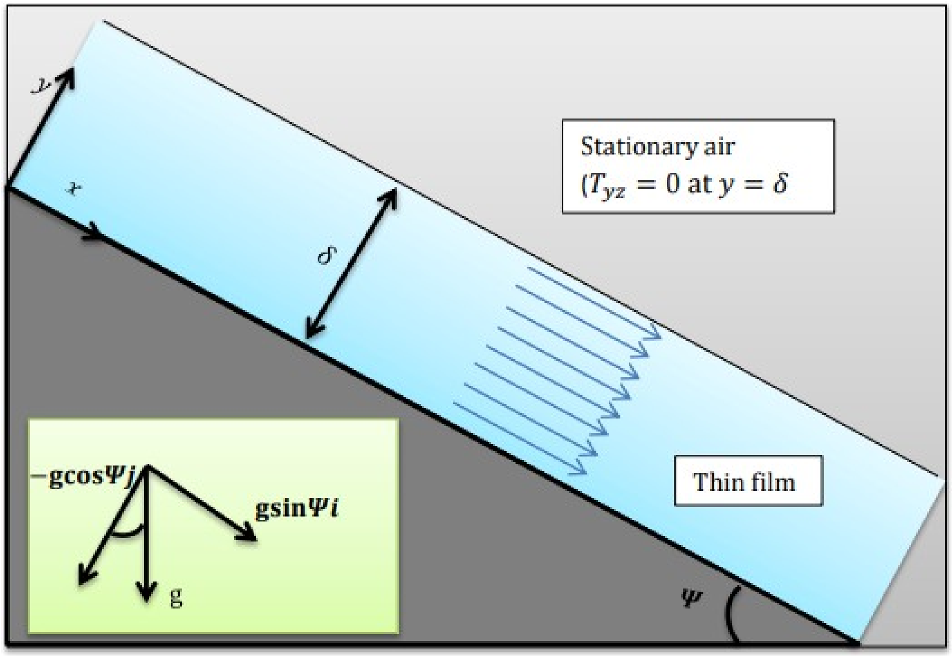
$$A_1=L+L^T.$$

Problem formulation

Let us assume a couple stress liquid flowing on an inclined surface in the form of thin film. The film is assumed to be of uniform thickness δ and the only force moving it is gravity. The geometry of the problem can be seen in figure 1. The assumptions on flow are

$$V=[u(y),0,0],\qquad \Psi=\Psi(y). \tag{4}$$

Figure 1: Geometry of the problem.



Source: from the authors (2025).

Eq. (1) is now satisfied exactly when using Eq. (4). Additionally, Eq. (2) is reduced to

$$\eta\frac{d^4u}{d\,y^4}-\mu\frac{d^2u}{dy^2}-\rho g\sin\theta=0, \tag{5}$$

The Eq. (3) becomes

$$\kappa\frac{d^2\Psi}{dy^2}+\mu\left(\frac{du}{dy}\right)^2=0. \tag{6}$$

The linked boundary conditions are

$$\text{At} \quad y=0, \qquad y=U, \qquad \frac{d^2u}{dx^2}=0, \qquad \Psi=0, \tag{7}$$

$$\text{At} \quad y=1, \qquad \frac{du}{dy}=0, \frac{d^3u}{dy^3}=0, \qquad \frac{d\Psi}{dy}=0. \tag{8}$$

For the non-dimensionalization of the Eq., the following non-dimensional parameters are presented.

$$y^{\square}=\frac{x}{\delta}, u^{\square}=\frac{u}{U}, \Psi^{\square}=\frac{\Psi-\Psi_0}{\Psi_1-\Psi_0}, \alpha^2=\frac{\mu\delta^2}{\eta}, k=\frac{\rho g\delta^4}{\eta U}\sin\theta, B_r=\frac{\mu U^2}{k\left(\Psi_1-\Psi_0\right)},$$

B_r represent the Brinkman number. Presenting these values in Eq. (5-8), and removing the "*" one gets

$$\frac{d^4u}{d\,y^4}-\alpha^2\frac{d^2u}{dy^2}=k, \tag{9}$$

$$\frac{d^2\Psi}{dy^2}=-B_r\left(\frac{du}{dy}\right)^2, \tag{10}$$

$$\text{At} \quad y=0, \qquad u=1, \qquad \frac{d^2u}{dy^2}=0, \qquad \Psi=0, \tag{11}$$

$$\text{At} \quad y=1, \qquad \frac{du}{dy}=0, \qquad \frac{d^3u}{dy^3}=0, \qquad \frac{d\Psi}{dy}=0. \tag{12}$$

Solutions of the problem

The results were obtained by employing a combination of analytical and numerical techniques. Initially, an exact solution was derived using mathematical manipulations and transformations. Subsequently, an approximate solution was developed using the Homotopy Perturbation Method (HPM), a semi-analytical technique that has been widely used to solve nonlinear differential equations. The HPM solution was obtained by implementing a Mathematica 13.0 code, which utilized numerical integration and algebraic manipulations to solve the nonlinear equations. The accuracy of the HPM solution was verified by comparing it with the exact solution, and the results showed excellent agreement.

Exact solution

The exact solution of Eq. (9) is

$$u=\frac{1}{2\left(1+{}^{2\alpha}\right)\alpha^4}\left({}^{-y\alpha}\left(\begin{matrix}2\;{}^{2\alpha}k-2\;{}^{y\alpha}k+2\;{}^{2y\alpha}k-2\;{}^{2\alpha+y\alpha}k\\+2\;{}^{y\alpha}ky\alpha^2+2\;{}^{2\alpha+y\alpha}ky\alpha^2-{}^{y\alpha}k\,y^2\alpha^2\\-{}^{2\alpha+y\alpha}k\,y^2\alpha^2+2\;{}^{y\alpha}\alpha^4+2\;{}^{2\alpha+y\alpha}\alpha^4\end{matrix}\right)\right).$$

Now, solving equation (10) with the appropriate boundary conditions Eq. (11) and Eq. (12) to get the following answer.

$$\Psi=\frac{1}{12\left(1+{}^{2\alpha}\right)^{2}\alpha^{8}}\,{}^{-2y\alpha}k^{2}\left(\begin{array}{c} -3\,{}^{4\alpha}+51\,{}^{2y\alpha}-48\,{}^{3y\alpha}-3\,{}^{4y\alpha}\\ -48\,{}^{(4+y)\alpha}-48\,{}^{(2+y)\alpha}+96\,{}^{2\alpha+2y\alpha}\\ -48\,{}^{(2+3y)\alpha}+51\,{}^{4\alpha+2y\alpha}+24\,{}^{2y\alpha}\alpha\\ -24\,{}^{3y\alpha}\alpha+24\,{}^{(2+y)\alpha}\alpha+24\,{}^{(4+y)\alpha}\alpha\\ -24\,{}^{(2+3y)\alpha}\alpha+24\,{}^{3y\alpha}y\alpha-24\,{}^{4\alpha+2y\alpha}\alpha\\ -24\,{}^{(2+y)\alpha}y\alpha-24\,{}^{(4+y)\alpha}y\alpha+24\,{}^{(2+3y)\alpha}y\alpha\\ -24\,{}^{2\alpha+2y\alpha}y\alpha^{2}+12\,{}^{2(1+y)\alpha}y^{2}\alpha^{2}+4\,{}^{2y\alpha}y\alpha^{4}\\ +8\,{}^{2\alpha+2y\alpha}y\alpha^{4}+4\,{}^{4\alpha+2y\alpha}y\alpha^{4}-6\,{}^{2y\alpha}y^{2}\alpha^{4}\\ -12\,{}^{2(1+y)\alpha}y^{2}\alpha^{4}-6\,{}^{2(2+y)\alpha}y^{2}\alpha^{4}+4\,{}^{2y\alpha}y^{3}\alpha^{4}\\ +8\,{}^{2(1+y)\alpha}y^{3}\alpha^{4}+4\,{}^{2(2+y)\alpha}y^{3}\alpha^{4}-\,{}^{2y\alpha}y^{4}\alpha^{4}\\ -2\,{}^{2(1+y)\alpha}y^{4}\alpha^{4}-\,{}^{2(2+y)\alpha}y^{4}\alpha^{4} \end{array}\right)B_{r}.$$

Now, the Homotopy Perturbation Method (HPM) is used to determine the approximate solutions to Eq. (9) and Eq. (10) subject to the boundary conditions Eq. (11) and Eq. (12). However, first we give the basics of the Homotopy Perturbation Method (HPM).

Basics of homotopy perturbation method (HPM)

When examining the non-linear differential equation with boundary conditions, the mathematical structure of HPM can be found (Siddiqui, Mahmood et al. 2006, Siddiqui, Mahmood et al. 2008).

$$C(j)=f(n);n\epsilon\Omega,K\left(j,\frac{\partial j}{\partial\delta}\right),n\epsilon I, \tag{13}$$

where $f(n)$ is an analytical function, K is the boundary operator, I denotes the domain boundary, and $C(j)$ is the differential operator. Assume we can rewrite Equation (13) as follows:

$$H(j)+N(j)=f(n). \tag{14}$$

Applying a Homotopy

$$\alpha(n,h):\Omega\times[0,1]\rightarrow M, \tag{15}$$

which satisfies

$$K\alpha(n,h)=(1-h), \tag{16}$$

where w_0 is the main estimate of Eq. (13), and $h\epsilon[0,1]$ is an insertion operator, which is thought of as a small parameter. Equation (16) series solution can be obtained by using the homology technique as follows.

$$v=v_0+hv_1+h^2v_2+\ldots, \tag{17}$$

If the approximate solution to equation (16) is reached using $h\rightarrow1$ equation (17), then

$$j(x)=\lim_{h\rightarrow1}v=v_0+hv_1+h^2v_2+\ldots. \tag{18}$$

Solution of the problem by HPM

Using HPM, the following component problems and their solutions in the momentum and energy equations are found. To obtain the solution, I employed the Homotopy Perturbation Method (HPM), a semi-analytical technique that involves constructing a homotopy equation. This equation is then used to decompose the solution into a series, which is solved recursively by solving a series of linear equations. By applying this approach, I was able to obtain an approximate analytical solution to the problem.

Zeroth component problems

The velocity and temperature of zeroth component equations Eq. (19) and Eq. (20), along with the associated boundary conditions Eq. (21) and Eq. (22), are

$$\frac{d^4 u_0}{dy^4} - k = 0, \tag{19}$$

$$\frac{d^2 \Psi_0}{dy^2} = 0, \tag{20}$$

$$\text{At} \quad y = 0, \quad u_0 = 1, \ \frac{d^2 u_0}{dy^2} = 0, \quad \Psi_0 = 0, \tag{21}$$

$$\text{At} \quad y = 1, \quad \frac{d u_0}{dy} = 0, \quad \frac{d^3 u_0}{dy^3} = 0, \quad \frac{d \Psi_0}{dy} = 0. \tag{22}$$

Solutions for zeroth component Eq. (19) and Eq. (20) along with the associated boundary conditions Eq. (21) and Eq. (22), are below in Eq. (23) and Eq. (24).

$$u_0(y) = \frac{1}{24}(24 + 8ky - 4ky^3 + ky^4). \tag{23}$$

$$\Psi_0(y) = 0. \tag{24}$$

First component problems

The velocity and temperature of first component equations Eq. (25) and Eq. (26), along with the associated boundary conditions Eq. (27) and Eq. (28), are

$$\frac{d^4 u_1}{dy^4} - \alpha^2 \left(\frac{d^2 u_0}{dy^2} \right) = 0, \tag{25}$$

$$\frac{d^2 \Psi_1}{dy^2} + B_r \left(\frac{d u_0}{dy} \right)^2 = 0, \tag{26}$$

$$\text{At} \quad y=0, \quad u_1=0, \ \frac{d^2u_1}{dy^2}=0, \quad \Psi_1=0, \tag{27}$$

$$\text{At} \quad y=1, \quad \frac{du_1}{dy}=0, \quad \frac{d^3u_1}{dy^3}=0, \quad \frac{d\Psi_1}{dy}=0. \tag{28}$$

Solutions for first component Eq. (25) and Eq. (26) along with the associated boundary conditions Eq. (27) and Eq. (28), are below in Eq. (29) and Eq. (30).

$$u_1(y)=\frac{1}{720}\left(-96ky\alpha^2+40ky^3\alpha^2-6ky^5\alpha^2+ky^6\alpha^2\right). \tag{29}$$

$$\Psi_1(y)=\frac{1}{10080}\begin{pmatrix} 544k^2yB_r-560k^2y^2B_r+280k^2y^4B_r \\ -56k^2y^5B_r-84k^2y^6B_r+40k^2y^7B_r-5k^2y^8B_r \end{pmatrix}. \tag{30}$$

Second component problems

The velocity and temperature of second component equations Eq. (31) and Eq. (32), along with the associated boundary conditions Eq. (33) and Eq. (34), are

$$\frac{d^4u_2}{dy^4}-\alpha^2\left(\frac{d^2u_1}{dy^2}\right)=0, \tag{31}$$

$$\frac{d^2\Psi_2}{dy^2}+2B_r\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)=0, \tag{32}$$

$$\text{At} \quad y=0, \quad u_2=0, \ \frac{d^2u_2}{dy^2}=0, \quad \Psi_2=0, \tag{33}$$

$$\text{At} \quad y=1, \quad \frac{du_2}{dy}=0, \quad \frac{d^3u_2}{dy^3}=0, \quad \frac{d\Psi_2}{dy}=0. \tag{34}$$

Solutions for second component Eq. (31) and Eq. (32) along with the associated boundary conditions Eq. (33) and Eq. (34), are below in Eq. (35) and Eq. (36).

$$u_2(y)=\frac{1}{40320}\left(2176ky\alpha^4-896ky^3\alpha^4+112ky^5\alpha^4-8ky^7\alpha^4+ky^8\alpha^4\right). \tag{35}$$

$$\Psi_2(y)=\frac{1}{907200}\begin{pmatrix} -39680k^2y\alpha^2B_r+40320k^2y^2\alpha^2B_r \\ -18480k^2y^4\alpha^2B_r+2016k^2y^5\alpha^2B_r \\ +5880k^2y^6\alpha^2B_r-1320k^2y^7\alpha^2B_r-675k^2y^8\alpha^2B_r+280k^2y^9\alpha^2B_r-28k^2y^{10}\alpha^2B_r \end{pmatrix}. \tag{36}$$

The second order solutions of velocity and temperature distribution are as below

$$u=u_0(y)+u_1(y)+u_2(y). \tag{37}$$

And

$$\Psi=\Psi_0(y)+\Psi_1(y)+\Psi_2(y). \tag{38}$$

For finding velocity, put Eq. (23), Eq. (29), and Eq. (35) in Eq. (37), the expression for velocity is

$$u=\frac{1}{24}\left(24+8ky-4ky^3+ky^4\right)+\frac{1}{720}\begin{pmatrix}-96ky\alpha^2+40ky^3\alpha^2\\ -6ky^5\alpha^2+ky^6\alpha^2\end{pmatrix}+\frac{1}{40320}\begin{pmatrix}2176ky\alpha^4-896ky^3\alpha^4\\ +112ky^5\alpha^4-8ky^7\alpha^4+ky^8\alpha^4\end{pmatrix}.$$
(39)

Now
For finding temperature distribution, put Eq. (24), Eq. (30), and Eq. (36) in Eq. (38), the expression for temperature is

$$\Psi=\frac{1}{10080}\begin{pmatrix}544k^2yB_r-560k^2y^2B_r\\ +280k^2y^4B_r\\ -56k^2y^5B_r-84k^2y^6B_r+40k^2y^7B_r-5k^2y^8B_r\end{pmatrix}+\frac{1}{907200}\begin{pmatrix}-39680k^2y\alpha^2B_r+40320k^2y^2\alpha^2 \\ -18480k^2y^4\alpha^2B_r+2016k^2y^5\alpha^2 \\ +5880k^2y^6\alpha^2B_r-1320k^2y^7\alpha^2 \\ -675k^2y^8\alpha^2B_r+280k^2y^9\alpha^2B_r-28k^2 \end{pmatrix}$$
(40)

Shear stress, vorticity, volumetric flow rate, average velocity

Shear stress on inclined surface

On inclined surface the shear stress is obtained by

$$T_{xy}\Big|_{y=0}=\mu\left(\frac{du}{dy}\right)\Big|_{y=0} \tag{41}$$

By substituting Eq. (39) in Eq. (41) we have

$$T_{xy}=\mu\left(\frac{k}{3}-\frac{2k\alpha^2}{15}+\frac{17k\alpha^4}{315}\right). \tag{42}$$

Vorticity

The vorticity of inclined is

$$\vec{\omega}=\nabla\times V=-\left(\frac{du}{dy}\right)\hat{k}. \tag{43}$$

By replacing the value of Eq. (39) in Eq. (43).we have

$$\bar{\omega}=-\hat{k}\left(\begin{array}{c}\frac{1}{24}\left(8k-12ky^2+4ky^3\right)+\frac{1}{720}\left(-96k\alpha^2+120ky^2\alpha^2\right.\\ \left.-30ky^4\alpha^2+6ky^5\alpha^2\right)+\frac{1}{40320}\left(2176k\alpha^4\right.\\ \left.-2688ky^2\alpha^4+560ky^4\alpha^4-56ky^6\alpha^4+8ky^7\alpha^4\right)\end{array}\right). \quad (44)$$

Volumetric flow rate

Inclined volumetric flow rate is determined by

$$Q=\int_0^1 u dy. \quad (45)$$

By putting the value of Eq. (39) in Eq. (45).we have

$$Q_{(HPM)}=1+\frac{2k}{15}-\frac{17k\alpha^2}{315}+\frac{62k\alpha^4}{2835} \quad (46)$$

Average velocity

For an inclined problem, the average velocity in dimensional form is provided by

$$\bar{u}=\frac{Q}{\delta}\ . \quad (47)$$

The average velocity and flow rate coincide in non-dimensional form, thus

$$\bar{u}_{(HPM)}=1+\frac{2k}{15}-\frac{17k\alpha^2}{315}+\frac{62k\alpha^4}{2835}. \quad (48)$$

Results and discussion

This paper studies the analytical solutions for modeling and simulations of a couple stress fluid flow and heat transfer in inclined geometry. Differential equations and related boundary conditions are developed in this problem. Non-linear differential equations are solved by using Homotopy Perturbation Method (HPM). The variation in temperature distribution and velocity profiles due to changed parameters likeα,k andB_r (Tables 1-4, Figures 2-8).

Table 1: Residual of velocity $\alpha=0.03, k=0.0005$,using HPM technique.

y	u_{HPM}	$Residual\, u_{HPM}$
0	1.	0.
0.1	1.00002	4.83978×10^{-15}
0.2	1.00003	9.55894×10^{-15}
0.3	1.00005	1.40403×10^{-14}
0.4	1.00006	1.8173×10^{-14}
0.5	1.00007	2.18558×10^{-14}
0.6	1.00008	2.49986×10^{-14}
0.7	1.00009	2.75252×10^{-14}
0.8	1.0001	2.93746×10^{-14}
0.9	1.0001	3.05023×10^{-14}
1	1.0001	3.08813×10^{-14}

Source: from the authors (2025).

Table 2: Residual of temperature for $\alpha=1.5$, $k=0.0007$, $B_r=0.01$, using HPM technique.

y	Ψ_{HPM}	$Residual\, \Psi_{HPM}$
0	0.	8.96016×10^{-10}
0.1	-1.96088×10^{-11}	8.72135×10^{-10}
0.2	-3.49849×10^{-11}	8.04376×10^{-10}
0.3	-4.64341×10^{-11}	7.01232×10^{-10}
0.4	-5.44345×10^{-11}	5.73945×10^{-10}
0.5	-5.95892×10^{-11}	5.73945×10^{-10}
0.6	-6.25666×10^{-11}	2.98979×10^{-10}
0.7	-6.40351×10^{-11}	1.77441×10^{-10}
0.8	-6.45968×10^{-11}	8.18953×10^{-11}
0.9	-6.47289×10^{-11}	2.09379×10^{-11}
1	-6.47378×10^{-11}	6.46105×10^{-25}

Source: from the authors (2025).

Table 3: Absolute difference for $\alpha=0.03$, $k=0.0005$, velocity profile and exact solution.

y	u_{HPM}	Exact u	Absolute difference
0	1.	1.	6.95×10^{-14}
0.1	1.00002	1.00002	7.0223×10^{-14}
0.2	1.00003	1.00003	7.69924×10^{-14}
0.3	1.00005	1.00005	1.32108×10^{-13}
0.4	1.00006	1.00006	1.13223×10^{-13}
0.5	1.00007	1.00007	8.43632×10^{-14}
0.6	1.00008	1.00008	7.92288×10^{-14}
0.7	1.00009	1.00009	3.45927×10^{-14}
0.8	1.0001	1.0001	2.38982×10^{-14}
0.9	1.0001	1.0001	4.78438×10^{-14}
1	1.0001	1.0001	1.59322×10^{-13}

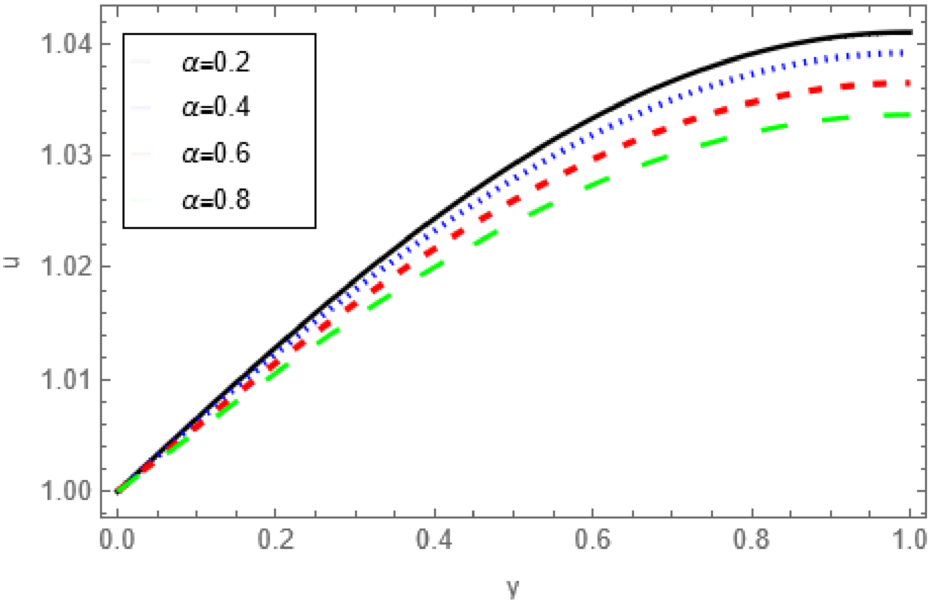
Source: from the authors (2025).

Table 4: Absolute difference for $\alpha=1.5$, $k=0.0007$, $B_r=0.01$, velocity profile and exact solution.

y	Ψ_{HPM}	Exact Ψ	Absolute difference
0	0.	1.30369×10^{-25}	0.
0.1	-1.96088×10^{-11}	6.47986×10^{-12}	2.60887×10^{-11}
0.2	-3.49849×10^{-11}	1.14934×10^{-11}	4.64783×10^{-11}
0.3	-4.64341×10^{-11}	1.51709×10^{-11}	6.1605×10^{-11}
0.4	-5.44345×10^{-11}	1.77014×10^{-11}	7.21359×10^{-11}
0.5	-5.95892×10^{-11}	1.9308×10^{-11}	7.88972×10^{-11}
0.6	-6.25666×10^{-11}	2.02239×10^{-11}	8.27905×10^{-11}
0.7	-6.40351×10^{-11}	2.06706×10^{-11}	8.47057×10^{-11}
0.8	-6.45968×10^{-11}	2.08399×10^{-11}	8.54367×10^{-11}
0.9	-6.47289×10^{-11}	2.08795×10^{-11}	8.56085×10^{-11}
1	-6.47378×10^{-11}	2.08822×10^{-11}	8.562×10^{-11}

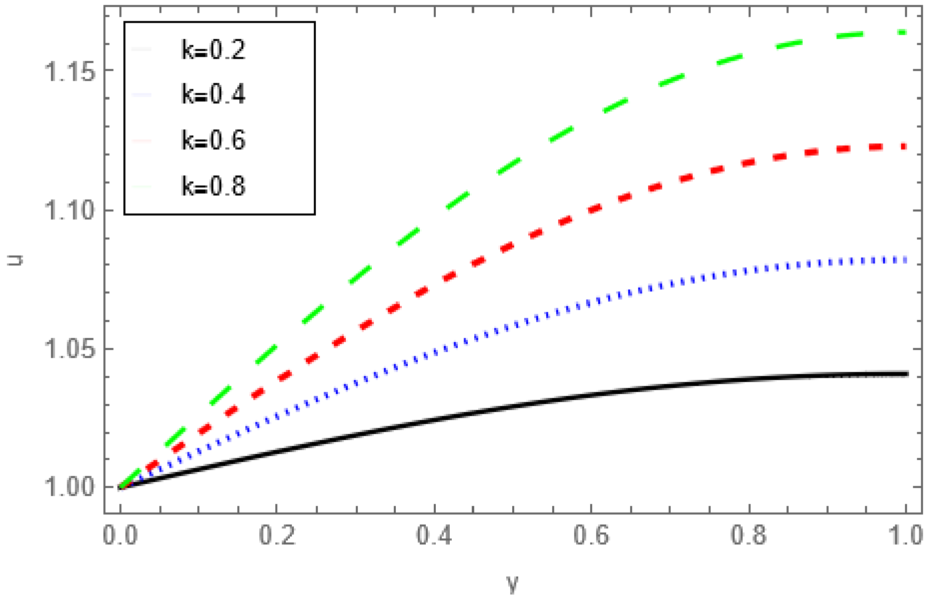
Source: from the authors (2025).

Figure 2: Velocity variation at different α keeping$k=0.2$.



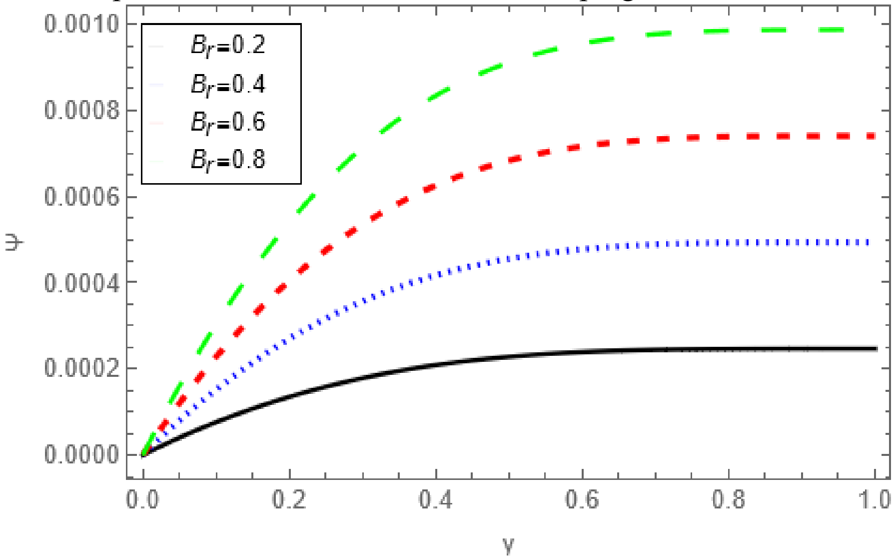
Source: from the authors (2025).

Figure 3: Velocity variation at different k keeping$\alpha=0.2$.



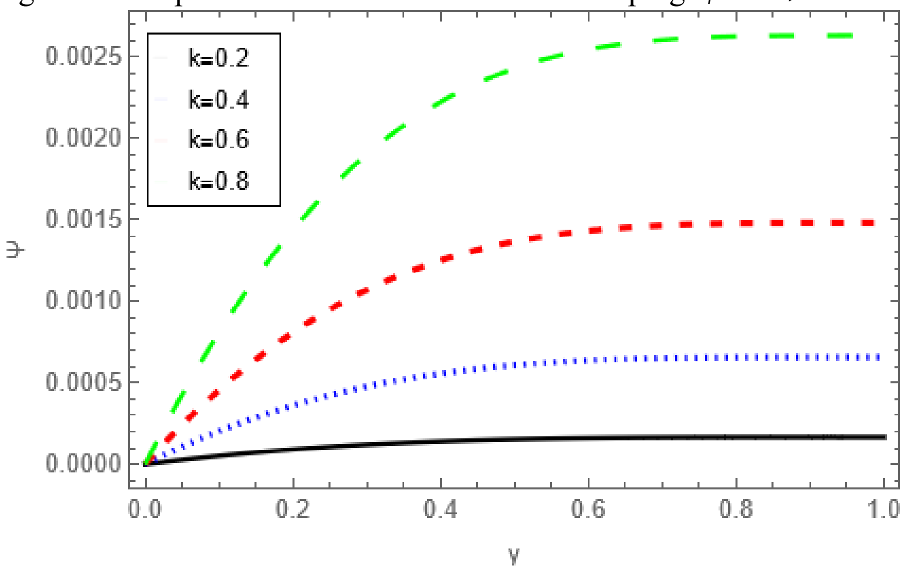
Source: from the authors (2025).

Figure 4: Temperature variation at different B_r keeping $B_r=0.2$, $k=0.3$, $\alpha=0.4$.



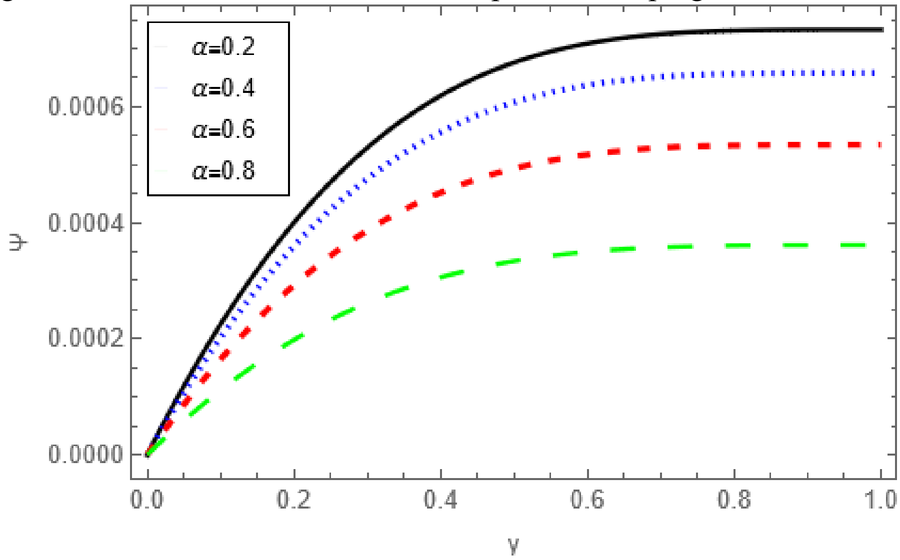
Source: from the authors (2025).

Figure 5: Temperature variation at different k keeping $B_r=0.3$, $\alpha=0.4$.



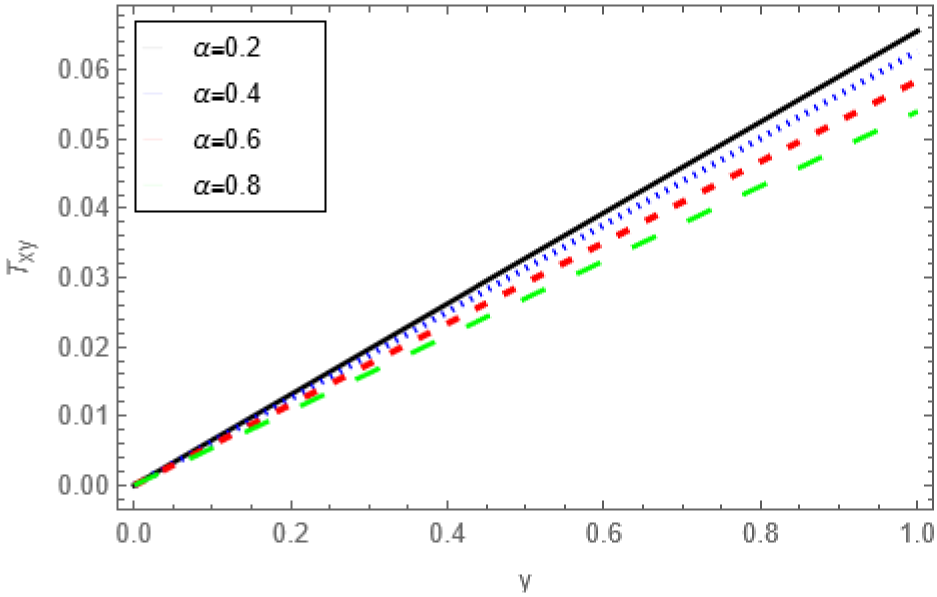
Source: from the authors (2025).

Figure 6: Variation at different α on temperature keeping $B_r = 0.3$, $k = 0.4$.



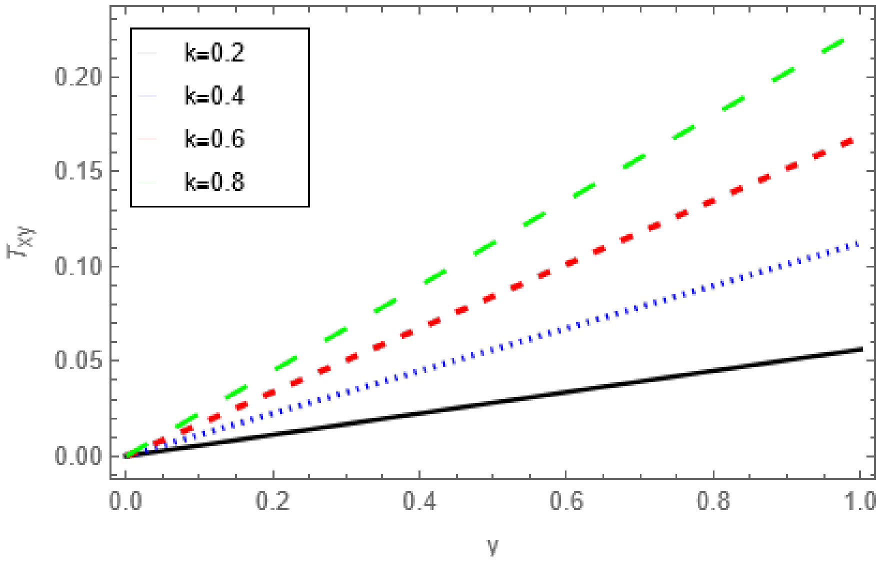
Source: from the authors (2025).

Figure 7: Variation of shear stress at different α keeping $k = 0.2$.



Source: from the authors (2025).

Figure 8: Variation of shear stress at different k keeping $\alpha = 0.7$.



Source: from the authors (2025).

The velocity profile and temperature distribution obtained by HPM, and the corresponding residual errors, are displayed in tables 1 and table 2. As the residuals are small enough, it guarantees the accuracy of the HPM solutions. Table 3 and table 4 show the absolute difference of exact and HPM solutions of velocity and temperature distribution. These tables also highlight the accuracy of the approximate solution obtained by the Homotopy Perturbation method. The impact of parameters α, and k on the velocity profile is displayed in figures 2 and 3. Figure 2 displays an inverse relationship between the velocity profile and α, while figure 3 illustrates a direct relationship between k and velocity. The impact of the parameters B_r, k and α on the temperature profiles is displayed in figures 4, 5, and 6. Figures 4 and 5 show a direct relationship between B_r, k and temperature profile, while figure 6 shows an inverse relationship between α and the temperature profile. The effect of factors α and k on shear stress is displayed in figures 7 and 8. Figure 7 displays an inverse relationship between shear stress and α, while figure 8 illustrates a direct relationship between k and shear stress.

Conclusion

This work has investigated the non-isothermal couple stress fluid's thin-film flow in an inclined plane using HPM. This method demonstrates the close relationship between the differential equations for temperature distributions, velocity, vorticity, shear stress, and volumetric flow rate. Both a mathematical and graphical calculation of the HPM results yields an amazing arrangement. The following is a list of the key conclusions.

- It's crucial to remember that normal stresses don't support a stable couple stress fluid flow.
- In the event that the average velocity, $\bar{u} > 0$, there will be a net upward flow.
- As parameter α increases, there is a corresponding drop in velocity.
- The increases in parameter k cause decrease in velocity.
- The temperature distribution grows as the Brinkman number B_r and k grow.
- The temperature distribution raises as the α raise.
- The increases in parameter k cause increase in shear stress.

- The upturns in parameter α cause drop in shear stress.

List of symbols

u Fluid velocity
L Gradient of V
η Couple Stress Parameter
Ψ Temperature
B_r Brinkman number
ρ Density
f Body force
P Pressure
C_p Specific heat
κ Thermal conductivity
$\frac{D}{Dt}$ Material time derivative
$\Psi^{\square}$ Dimensionless temperature
$u^{\square}$ Dimensionless velocity
A_1 First Rivilin–Erickson tensor

Declarations

Conflict of Interest: The authors declare no conflict of interest.

Funding: No external funding received regarding this research.

Availability of data and materials: Data will be provided on request to the corresponding author.

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