ISSN: 2317-0840

Analysis of the centroid as a critical point of the harmonic mean function in triangles

Luiz Cláudio Pereira[1†], Rogério César dos Santos[1],
Ruthyelen Cristina Machado de Freitas[2].

[1] *Universidade de Brasília; Faculdade UnB Planaltina;Brasília-DF, Brasil.*
[2] *Secretaria de Educação do Distrito Federal; Brasília-DF, Brasil.*

Abstract: *Inspired by previous works, considering an arbitrary triangle ABC, where AM is the median relative to side BC and $P \in AM$, in this article, we propose to investigate the behavior of the harmonic mean function of PB^2 and PC^2. By analyzing the critical points of this function, we show that the relative extrema coincides with the centroid of the triangle if and only if the sides of the triangle are roots of a certain homogeneous polynomial of degree six. We also present some numerical simulations to illustrate the study conducted. The study is relevant as it brings previously unknown properties of the centroid of a triangle and has the potential for further investigations by considering other functions besides the harmonic mean between PB^2 and PC^2. In this work, we use differential calculus, the law of cosines, Stewart's theorem, and the GeoGebra software.*

Keywords: *Harmonic mean; centroid; relative extrema.*

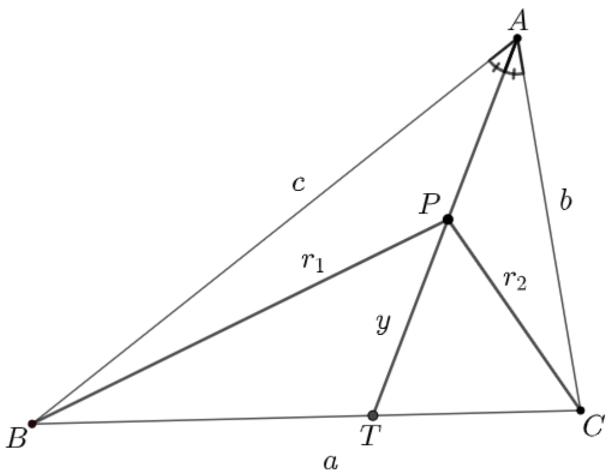
Introduction

Bialostocki and Bialostocki (2011) have shown that given a triangle ABC with a point P on the bisector of the angle $\angle BAC$, as illutrated in figure 1, the extremal values of

$$\frac{r_1}{r_2} = \frac{PB}{PC}, \tag{1}$$

occur at the incenter and the excenter on the opposite side of A.

Figure 1: Point P on the angle bisector AT.



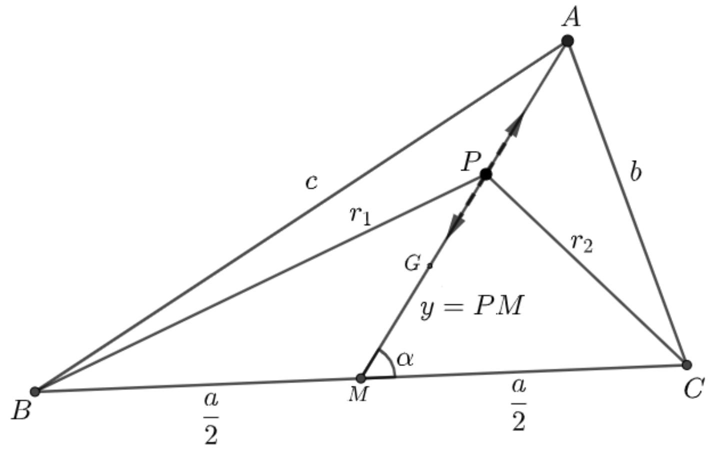
Source: figure created by authors using GeoGebra.

In additional articles, using different cevians from the point A, Hajja (2012), Bialostocki and Ely (2015) and Hajja (2017) have proven more general results and pointed out some observations about the ratio (1).

†Corresponding author: cpereira@unb.br
Manuscript received: 19/12/2024; Revised: 28/02/2025; Accepted: 14/03/2025

On the other hand, as illustrated in figure 2, using the median rather than the bisector, Santos, Freitas and Velasco (2024) have shown that, as a point P moves on AM, it is not always true that the centroid optimizes the ratio (1).

Figure 2: centroid G as a possible solution for an optimization problem.



Source: figure created by authors using Geogebra.

The article by Santos, Freitas and Velasco (2024) simply states that the centroid G of the triangle is the relative minimum of the ratio (1) if and only if

$$5a^2 = b^2 + c^2. \tag{2}$$

In another work, using the median, Santos and Freitas (2024) have demonstrated that a necessary and sufficient condition for the centroid to be the relative minimum of the geometric mean

$$\sqrt{r_1 \cdot r_2}, \tag{3}$$

is that

$$4a^4 - 7a^2(b^2 + c^2) + [7(b^4 + c^4) - 22b^2c^2] = 0. \tag{4}$$

In this article, following these ideas, using the median AM, we study the extrema points of the harmonic mean given by

$$\frac{2}{\dfrac{1}{PB^2} + \dfrac{1}{PC^2}},$$

and investigate the condition to the point P to coincide with the centroid G.

What motivates us in our work is the possibility of delving deeper into the study of Plane Geometry, bringing new properties of the centroid of a triangle, characterizing triangles whose centroid is the extreme of the aforementioned function.

Critical points of the harmonic mean function

In order to proceed, consider any triangle ABC and let be

$$BC = a, \quad AC = b, \quad AB = c \quad \text{and} \quad M \text{ the midpoint of the side } BC. \tag{5}$$

By definition, the straight line AM is the median relative to the side BC.

Suppose that P is any point on AM. Assume that

$$y = PM, \quad PB = r_1(y), \quad PC = r_2(y), \quad \alpha = \angle AMC. \tag{6}$$

By symmetry, without loss of generality, we can restrict ourselves to

$$\alpha = \angle AMC \in \left(0, \frac{\pi}{2}\right].$$

It will be seen that the harmonic mean function of the squares of PB and PC

$$\frac{2}{\dfrac{1}{PB^2} + \dfrac{1}{PC^2}} = \frac{2}{\dfrac{1}{r_1^2(y)} + \dfrac{1}{r_2^2(y)}},$$

has critical points at

$$y_0 = 0, \quad y_1 = \frac{a\sqrt{-1 + 2\cos\alpha}}{2}, \quad y_2 = -y_1.$$

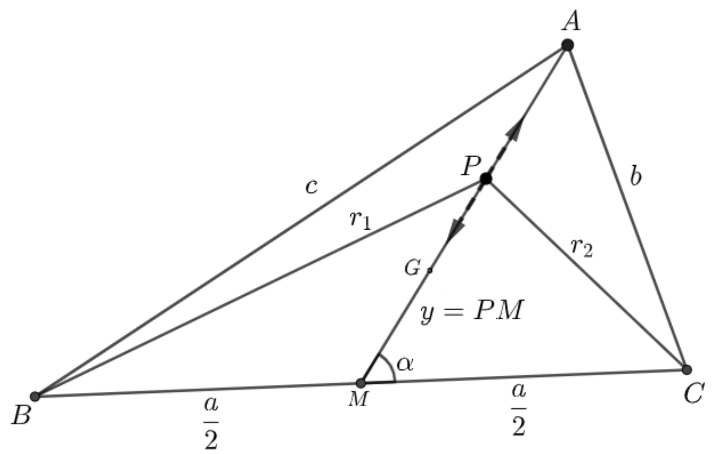
In addition, positive critical point coincides with the centre of gravity of the triangle ABC if and only if the following expression involving the sides of the triangle

$$-8a^6 + 12\left(b^2 + c^2\right)a^4 + 3\left(-11b^4 + 32b^2c^2 - 11c^4\right)a^2 + 3\left(b^4c^2 + b^2c^4\right) + b^6 + c^6 = 0,$$

is true.

Now, let us prove these results. Figure 3 illustrates.

Figure 3: Proving the results.



Source: figure created by authors using Geogebra.

First, notice that if we apply the law of cosines to the triangles PMB and PMC, we get

$$\begin{aligned} r_1^2(y) &= y^2 + \frac{a^2}{4} - 2y\frac{a}{2}\cos(\pi - \alpha) \\ &= y^2 + \frac{a^2}{4} + ya\cos\alpha, \end{aligned}$$

and

$$\begin{aligned} r_2^2(y) &= y^2 + \frac{a^2}{4} - 2y\frac{a}{2}\cos\alpha \\ &= y^2 + \frac{a^2}{4} - ya\cos\alpha. \end{aligned}$$

Therefore, the harmonic mean f is given by one of the following forms

$$\begin{aligned}
f(y) &= \frac{2}{\dfrac{1}{PB^2}+\dfrac{1}{PC^2}} \\
&= \frac{2}{\dfrac{1}{r_1^2(y)}+\dfrac{1}{r_2^2(y)}} \\
&= \frac{2r_1^2(y)r_2^2(y)}{r_1^2(y)+r_2^2(y)} \\
&= \frac{2\left(y^2+\dfrac{a^2}{4}+ya\cos\alpha\right)\left(y^2+\dfrac{a^2}{4}-ya\cos\alpha\right)}{y^2+\dfrac{a^2}{4}+ya\cos\alpha+y^2+\dfrac{a^2}{4}-ya\cos\alpha} \\
&= \frac{2\left(y^2+\dfrac{a^2}{4}+ya\cos\alpha\right)\left(y^2+\dfrac{a^2}{4}-ya\cos\alpha\right)}{2y^2+\dfrac{a^2}{2}} \qquad (7) \\
&= \frac{4y^4+4a^2\left(\dfrac{1}{2}-\cos^2\alpha\right)y^2+\dfrac{a^4}{4}}{4y^2+a^2} \qquad (8) \\
&= \frac{4y^4-2a^2\cos(2\alpha)\,y^2+\dfrac{a^4}{4}}{4y^2+a^2}, \qquad (9)
\end{aligned}$$

since

$$\begin{aligned}
\cos(2\alpha)=\cos^2\alpha-\sin^2\alpha \quad &\Leftrightarrow \quad \cos(2\alpha)=\cos^2\alpha-\left(1-\cos^2\alpha\right) \\
&\Leftrightarrow \quad \cos(2\alpha)=2\cos^2\alpha-1 \\
&\Leftrightarrow \quad \frac{1}{2}-\cos^2\alpha=-\frac{\cos(2\alpha)}{2}.
\end{aligned}$$

For the purposes of calculation, it is convenient to set

$$n:=a\cos\alpha \quad \text{and} \quad m:=\frac{a^2}{4}. \qquad (10)$$

Thus, the expression (7) yields

$$f(y)=\frac{\left[(y^2+m)+yn\right]\left[(y^2+m)-yn\right]}{y^2+m}. \qquad (11)$$

From the difference of two squares given in the numerator of (11), we obtain

$$\begin{aligned}
f(y) &= \frac{(y^2+m)^2}{y^2+m}-\frac{n^2y^2}{y^2+m} \\
&= y^2+m-\frac{n^2y^2}{y^2+m}. \qquad (12)
\end{aligned}$$

By differentiation, we get

$$\begin{aligned} f'(y) &= 2y - \frac{2yn^2(y^2+m) - 2y^3n^2}{(y^2+m)^2} \\ &= 2y\left[1 - \frac{n^2\left(y^2+m\right) - y^2n^2}{\left(y^2+m\right)^2}\right] \\ &= \frac{2y\left[\left(y^2+m\right)^2 - n^2m\right]}{\left(y^2+m\right)^2}, \end{aligned} \tag{13}$$

whence it follows that the critical points for the harmonic mean f are such that

$$y = 0, \tag{14}$$

or

$$y^4 + 2my^2 + m^2 - n^2m = 0. \tag{15}$$

Since (15) is a biquadratic equation, it is a quadratic equation with respect to y^2,

$$\left(y^2\right)^2 + 2m\left(y^2\right) + \left(m^2 - n^2m\right) = 0. \tag{16}$$

As a consequence, we have

$$\begin{aligned} y^2 &= \frac{-2m \pm \sqrt{4m^2 - 4m^2 + 4n^2m}}{2} \\ &= -m \pm n\sqrt{m}. \end{aligned}$$

Hence, from (10), we deduce the harmonic mean f has also critical points at

$$\begin{aligned} y &= \pm\sqrt{-m \pm n\sqrt{m}} \\ &= \pm\sqrt{-\frac{a^2}{4} \pm (a\cos\alpha)\frac{a}{2}} \\ &= \pm\frac{a}{2}\sqrt{-1 \pm 2\cos\alpha}, \end{aligned} \tag{17}$$

besides $y = 0$.

Since by symmetry, we can restrict ourselves to

$$\alpha = \angle AMC \in \left]0, \frac{\pi}{2}\right],$$

it is obvious that $\cos\alpha > 0$ and

$$-1 - 2\cos\alpha < -1 - 2\cdot 0 < -1.$$

Furthermore,

$$\begin{aligned} -1 + 2\cos\alpha \geq 0 \quad &\Leftrightarrow \quad \cos\alpha \geq \frac{1}{2} \\ &\Leftrightarrow \quad \alpha \leq \frac{\pi}{3}. \end{aligned}$$

These calculations show that the three critical points for the harmonic mean function f are

$$y_0 = 0, \quad \text{for every } \alpha \in \left]0, \frac{\pi}{2}\right], \tag{18}$$

and

$$y_1 = \frac{a\sqrt{-1+2\cos\alpha}}{2} \quad \text{and} \quad y_2 = -y_1, \quad \text{for every } \alpha \in \left]0, \frac{\pi}{3}\right]. \tag{19}$$

In order to study the relative extrema on the domain of the harmonic mean function, we expand on the numerator of (13) to get

$$\begin{aligned} f'(y) &= \frac{2\left[y^5 + 2my^3 + \left(m^2 - n^2 m\right) y\right]}{(y^2+m)^2} \\ &= \frac{2\left(y^5 + 2my^3 + ty\right)}{(y^2+m)^2}, \end{aligned}$$

where, for convenience, we set

$$t := m^2 - n^2 m. \tag{20}$$

Perfoming straightforward calculations, we obtain that the second derivative of f is given by

$$\begin{aligned} f''(y) &= 2\left[\frac{(5y^4 + 6my^2 + t)(y^2+m)^2 - 4\left(y^6 + 2my^4 + ty^2\right)(y^2+m)}{(y^2+m)^4}\right] \\ &= 2\left[\frac{(5y^4 + 6my^2 + t)(y^2+m) - 4(y^6 + 2my^4 + ty^2)}{(y^2+m)^3}\right] \\ &= 2\left[\frac{y^6 + 3my^4 + 3\left(2m^2 - t\right)y^2 + tm}{(y^2+m)^3}\right] \\ &= 2\left[\frac{y^6 + 3my^4 + 3\left(m^2 + n^2 m\right)y^2 + tm}{(y^2+m)^3}\right]. \end{aligned} \tag{21}$$

Now, from (10) and (20), we notice that

$$\begin{aligned} t &= \left(\frac{a^2}{4}\right)^2 - (a\cos\alpha)^2\,\frac{a^2}{4} \\ &= \frac{a^4}{4}\left[\frac{1}{4} - (a\cos\alpha)^2\right] \\ &= \frac{a^4}{4}\left(\frac{1}{2} + \cos\alpha\right)\left(\frac{1}{2} - \cos\alpha\right). \end{aligned}$$

Hence, for every $\alpha \in \left]0, \frac{\pi}{3}\right[$, one has

$$\cos\alpha \in \left]\frac{1}{2}, 1\right[\quad \Leftrightarrow \quad \frac{1}{2} < \cos\alpha < 1,$$

which implies $t < 0$, and it holds that

$$f''(y_0) \quad = \quad f''(0) \quad = \quad \frac{2t}{m^2} \quad < \quad 0. \tag{22}$$

Similarly, for every $\alpha \in \left]\frac{\pi}{3}, \frac{\pi}{2}\right]$, we have

$$\cos\alpha \in \left[0, \frac{1}{2}\right[\quad \Leftrightarrow \quad 0 \leq \cos\alpha < \frac{1}{2},$$

which implies $t > 0$, and it is true that

$$f''(y_0) \quad = \quad f''(0) \quad = \quad \frac{2t}{m^2} \quad > \quad 0, \tag{23}$$

Of course, the above results show that $y_0 = 0$ is the relative minimum point of the harmonic mean for α in the interval

$$\left]\frac{\pi}{3}, \frac{\pi}{2}\right],$$

and that $y_0 = 0$ is the relative maximum point of harmonic mean for α in the interval

$$\left]0, \frac{\pi}{3}\right[.$$

To investigate the other two critical points given in (19), notice that the harmonic mean f is an even function since it follows from (9) that $f(-y) = f(y)$.

Moreover, this investigation requires of us we keep in mind that $\alpha \in \left]0, \frac{\pi}{3}\right]$.

Under the circumstances, one ought to expect that

$$y_1 = \frac{a\sqrt{-1 + 2\cos\alpha}}{2} \quad \text{and} \quad y_2 = -y_1,$$

are both relative minimum points of the harmonic mean.

To verify this is the case, let us define

$$s := -m + n\sqrt{m}, \tag{24}$$

and observe that from (10) we have

$$\begin{aligned} s &= -\frac{a^2}{4} + (a\cos\alpha)\sqrt{\frac{a^2}{4}} \\ &= \frac{a^2}{4}(-1 + 2\cos\alpha) \\ &= (y_1)^2. \end{aligned} \tag{25}$$

Consequently, in addition to zero, for every $\alpha \in \left]0, \frac{\pi}{3}\right]$, the other two critical points of the harmonic mean are $\sqrt{s}$ and $-\sqrt{s}$.

According to (20) and (24), we also note that

$$s + m = n\sqrt{m},$$

and

$$t - m^2 = -n^2 m.$$

Using all these remarks, it is easy to compute from (10), (20), (21), (24) and (25) that

$$\begin{aligned} f''\left(\sqrt{s}\right) &= f''\left(-\sqrt{s}\right) \\ &= 2\left[\frac{s^3+3ms^2+3\left(m^2+n^2m\right)s+tm}{(s+m)^3}\right] \\ &= 2\left[\frac{(s+m)^3-m^3+3n^2ms+tm}{n^3m\sqrt{m}}\right] \\ &= 2\left[\frac{(s+m)^3+\left(t-m^2\right)m+3n^2ms}{n^3m\sqrt{m}}\right] \end{aligned} \tag{26}$$

$$\begin{aligned} &= 2\left[\frac{(n\sqrt{m})^3+\left(-n^2m\right)m+3n^2ms}{n^3m\sqrt{m}}\right] \\ &= 2\left[\frac{n\sqrt{m}-m+3s}{n\sqrt{m}}\right] \\ &= 2\left[\frac{s+3s}{n\sqrt{m}}\right] \\ &= \frac{8s}{n\sqrt{m}} \\ &= \frac{2a^2\left(-1+2\cos\alpha\right)}{(a\cos\alpha)\sqrt{\dfrac{a^2}{4}}} \\ &= \frac{4\left(-1+2\cos\alpha\right)}{\cos\alpha}. \end{aligned} \tag{27}$$

Since $\alpha \in \left]0, \dfrac{\pi}{3}\right[$, we obtain

$$\frac{1}{2} < \cos\alpha < 1 \quad \Leftrightarrow \quad 1 < 2\cos\alpha < 2,$$

and, as a consequence, it yields

$$f''\left(\sqrt{s}\right) > 0. \tag{28}$$

Therefore, we conclude that

$$y_1 = \frac{a\sqrt{-1+2\cos\alpha}}{2} \quad \text{and} \quad y_2 = -y_1, \quad \text{for every } \alpha \in \left]0, \frac{\pi}{3}\right[, \tag{29}$$

are both relative minimum points of the harmonic media f.

In the next section, we shall determine what condition must be verified in order the point P to be the relative extremum point of the harmonic mean, and to coincide with the centre of gravity of the triangle ABC.

Centroid and harmonic mean

Consider any triangle ABC whose straight line AM is the median relative to the side BC.

On applying law of cosines to the triangles AMC and AMB, it follows that

$$b^2 = AM^2 + \frac{a^2}{4} - a \cdot AM \cdot \cos\alpha, \tag{30}$$

and

$$c^2 = AM^2 + \frac{a^2}{4} - a \cdot AM \cdot \cos(\pi - \alpha) \quad \Leftrightarrow \quad c^2 = AM^2 + \frac{a^2}{4} + a \cdot AM \cdot \cos\alpha, \tag{31}$$

where M is the midpoint of the side BC,

$$a = BC, \quad b = AC, \quad c = AB, \quad \text{and} \quad \alpha = \angle AMC.$$

From expressions (30) and (31), we obtain the Stewart's theorem applied to the median,

$$AM^2 = \frac{2\left(b^2 + c^2\right) - a^2}{4}, \tag{32}$$

and

$$2a \cdot AM \cdot \cos\alpha = c^2 - b^2 \quad \Leftrightarrow \quad \cos\alpha = \frac{c^2 - b^2}{a\sqrt{2\left(b^2 + c^2\right) - a^2}}. \tag{33}$$

Now, according to last section, relative extrema points of the harmonic mean

$$\frac{2}{\dfrac{1}{PB^2} + \dfrac{1}{PC^2}}, \tag{34}$$

are given by

$$PM = 0, \quad PM = m_0 \quad \text{and} \quad PM = -m_0, \tag{35}$$

where

$$m_0 = \frac{a\sqrt{-1 + 2\cos\alpha}}{2}, \tag{36}$$

is the relative minimum of (34). The relative maximum of the harmonic mean occurs at zero.

Furthermore, since harmonic mean as a function of

$$y = PM,$$

is an even one, it follows that $-m_0$ is also a relative minimum point of it.

As a result, we can make our reasoning using only m_0.

In this way, for the point P to be the centroid of the triangle ABC, it is necessary and sufficient that

$$PM = \frac{AM}{3}, \tag{37}$$

since it is known that center of gravity of a triangle lies two-thirds of the distance from the vertex to the opposite side along a median.

Therefore, the statement that P is a relative extremum point of the harmonic mean, and centroid of the triangle ABC is equivalent to requiring that

$$m_0 = \frac{AM}{3}$$

$$\Leftrightarrow 3m_0 = AM \Leftrightarrow 9m_0^2 = AM^2$$

$$\Leftrightarrow 9a^2\left(-1+2\cos\alpha\right)=2\left(b^2+c^2\right)-a^2$$

$$\Leftrightarrow 9a^2\cos\alpha=b^2+c^2+4a^2$$

$$\Leftrightarrow 81a^4\cos^2\alpha=\left(4a^2+b^2+c^2\right)^2$$

$$\Leftrightarrow \frac{81a^2\left(c^2-b^2\right)^2}{2\left(b^2+c^2\right)-a^2}=\left(4a^2+b^2+c^2\right)^2$$

$$\Leftrightarrow 81a^2\left(c^2-b^2\right)^2=\left[2\left(b^2+c^2\right)-a^2\right]\left(4a^2+b^2+c^2\right)^2$$

$$\begin{aligned}
\Leftrightarrow\quad & 81a^2\left(c^2-b^2\right)^2+a^2\left(4a^2+b^2+c^2\right)^2=2\left(b^2+c^2\right)\left(4a^2+b^2+c^2\right)^2\\
\Leftrightarrow\quad & 8a^6+4a^4b^2+4a^4c^2+41a^2b^4-80a^2b^2c^2+41a^2c^4=\\
& \qquad b^6+c^6+16a^4b^2+16a^4c^2+8a^2b^4+8a^2c^4+16a^2b^2c^2+3b^4c^2+3b^2c^4\\
\Leftrightarrow\quad & -8a^6+12a^4\left(b^2+c^2\right)+3a^2\left(-11b^2+32b^2c^2-11c^4\right)\\
& \qquad\qquad +3\left(b^4c^2+b^2c^4\right)+b^6+c^6=0.
\end{aligned}$$

Consequently, we have proven that relative minimum points of the harmonic mean

$$\frac{2}{\dfrac{1}{PB^2}+\dfrac{1}{PC^2}},$$

coincides with the centroid of the triangle ABC, or its symmetrical point with respect to midpoint of side BC if and only if it holds the expression

$$-8a^6+12\left(b^2+c^2\right)a^4+3\left(-11b^4+32b^2c^2-11c^4\right)a^2+3\left(b^4c^2+b^2c^4\right)+b^6+c^6=0, \qquad (38)$$

where

$$a=BC,\quad b=AC\quad \text{and}\quad c=AB. \qquad (39)$$

Numerical simulations

It is worthwhile to observe that if we define the polynomial

$$\phi\left(a,b,c\right)=-8a^6+12\left(b^2+c^2\right)a^4+3\left(-11b^4+32b^2c^2-11c^4\right)a^2+3\left(b^4c^2+b^2c^4\right)+b^6+c^6,$$

then

$$\phi(a,b,c)=\phi(a,c,b).$$

This result might be expected from the beginning because of the restriction

$$\alpha=\angle AMC\in\left(0,\frac{\pi}{2}\right].$$

Another interesting property of this polynomial ϕ is that it is homogeneous of degree six since

$$\phi\left(ka,kb,kc\right)=k^6\phi\left(a,b,c\right),\quad \text{for every}\quad k\in\mathbb{R}\quad \text{and}\quad (a,b,c)\in\mathbb{R}^3.$$

For numerical simulation of the main result obtained in preceding section, we must find triples $(a,b,c) \in \mathbb{R}^3$ in the first octant in three dimensional space such that

$$\phi\,(a,b,c) = 0,$$

and all following existence conditions hold

$$0 < a < b + c, \quad 0 < b < a + c, \quad \text{and} \quad 0 < c < a + b. \tag{40}$$

Now, as discussed in an early section, to proceed we need to separate our study into two cases.

First case: $0 < \alpha < \dfrac{\pi}{3}$

To begin with, let us observe that we can perform most of the previous calculations with algebraic software, like Maxima or Maple. This adds efficiency and speeds up the task of obtaining the results shown.

In addition, it provides an elegant alternative to achieve triples $(a,b,c) \in \mathbb{R}^3$ in the first octant such that $\phi\,(a,b,c) = 0$ and satisfying the existence conditions (40).

For a specific example, let us assign

$$a := 3 \quad \text{and} \quad c := 4.$$

From (38), utilizing algebraic software, like Maxima or Maple, we have

$$b^6 - 249b^4 + 15564b^2 - 62216 = 0. \tag{41}$$

Using software, it is readily shown that the solution set of (41) is

$$\left\{\sqrt{101}, -\sqrt{101}, \sqrt{74 - 18\sqrt{15}}, -\sqrt{74 - 18\sqrt{15}}, \sqrt{74 + 18\sqrt{15}}, -\sqrt{74 + 18\sqrt{15}}\right\}.$$

Now it can be seen that the unique solution of (41) satisfying the existence conditions (40) is given by

$$b = \sqrt{74 - 18\sqrt{15}}.$$

The following Maple code can be used to accomplish the above task.

```
restart;
with(Student);
N:=(a,b,c)->c^6+(-33*a^2+3*b^2)*c^4+(12*a^4+96*a^2*b^2+3*b^4)*c^2
                -8*a^6+12*a^4*b^2-33*a^2*b^4+b^6;
S:=b->N(3,b,4);
solve(S(b),b);
evalf(solve(S(b),b));
```

In conclusion, all these numerical results ensure that if we assume

$$a = 3, \quad b = \sqrt{74 - 18\sqrt{15}} \approx 2.070338081, \quad \text{and} \quad c = 4,$$

then, by a direct calculation, it holds

$$\phi\left(3, \sqrt{74 - 18\sqrt{15}}, 4\right) = 0,$$

and there exists a triangle ABC such that

$$BC = a = 3, \quad AC = b = \sqrt{74 - 18\sqrt{15}} \quad \text{and} \quad AB = c = 4, \tag{42}$$

since triangle inequalities (40) are satisfied.

In this case, from expression (33), we obtain

$$\cos\alpha = \frac{-58 + 18\sqrt{15}}{3\sqrt{171 - 36\sqrt{15}}} \quad \Leftrightarrow \quad \cos\alpha \approx 0.6948925897$$

$$\Leftrightarrow \quad \alpha \approx 0.8025258016 \quad \text{radians}$$

$$\Leftrightarrow \quad \alpha \approx 45.98134137 \quad \text{degrees.}$$

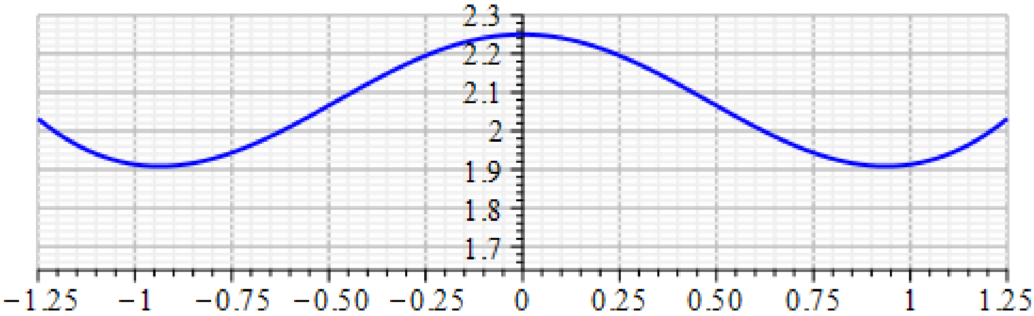
From expression (36), the relative minimum of the harmonic mean occurs at

$$m_0 \approx 0.9364916725 \quad \text{and} \quad -m_0.$$

The figure below illustrates the behaviour of the harmonic mean for the numerical values of the sides of triangle ABC given in (42).

We used Maple, but we could have used any other mathematical graphing software.

Figure 4: Graphic of the harmonic mean as function of $y = PM$.



Source: figure created by authors using Maple.

As might be expected from previous calculations, it is clear from this graph that the relative maximum value of the harmonic mean is $2,25$ at $y = 0$.

Since we have the numerical values of the sides of the triangle ABC, it is possible to draw it using Geogebra, for instance.

On using this program, the triangle ABC given below has been created from the beginning with only numerical values of its sides given in (42).

As consequence, one can use the program to verify the value of

$$\alpha = \angle AMC \approx 45.98134 \text{ degrees,}$$

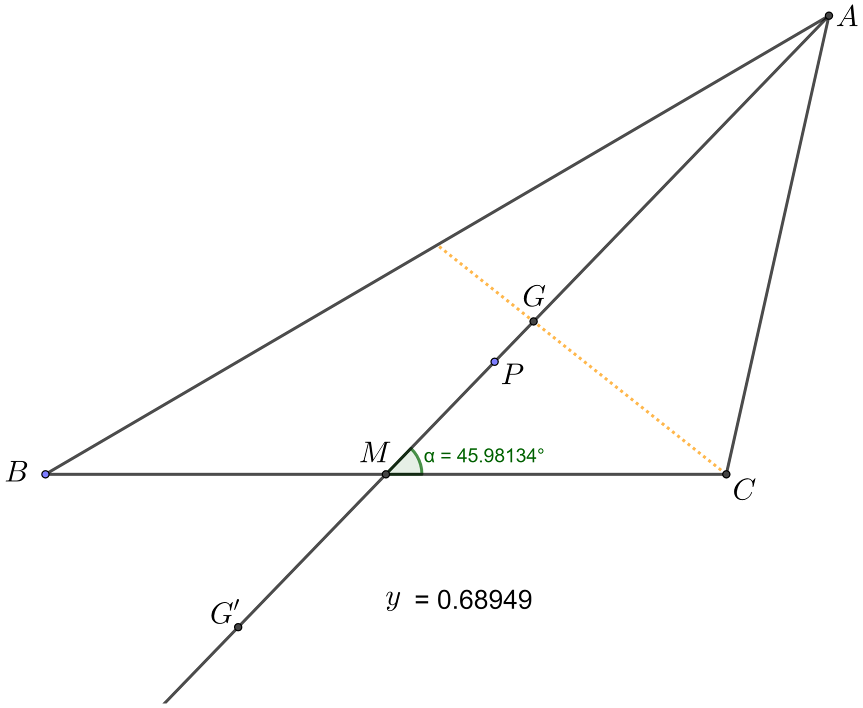
and to measure dynamically the distance from P to M by moving point P over the straight line defined by A and M.

Also, this allows to see that when P coincides with the centre of gravity G of the triangle ABC or its symmetrical point G', with respect to the midpoint of the side BC, the distance from P to M equals

$$m_0 \approx 0.94.$$

An interactive dynamic simulation of the triangle ABC given in Figure 5 can be visualized from the hyperlink, moving point P < here > .

Figure 5: Triangle ABC with $BC = 3$, $AC = \sqrt{74 - 18\sqrt{15}}$ and $AB = 4$.



Source: figure created by authors using Geogebra.

Above picture shows point P located at a position in which

$$y = PM \approx 0.68949.$$

Second case: $\dfrac{\pi}{3} \leq \alpha \leq \dfrac{\pi}{2}$

Now, we assume that

$$a = \frac{\sqrt{41 + \sqrt{1033 - 324\sqrt{3}}}}{5} \approx 1.5839357979, \quad b = \frac{4}{5}, \quad \text{and} \quad c = 1.$$

Triangle inequalities (40) are satisfied. Hence, there exists a triangle ABC such that

$$BC = a = \frac{\sqrt{41 + \sqrt{1033 - 324\sqrt{3}}}}{5}, \quad AC = b = \frac{4}{5} \quad \text{and} \quad AB = c = 1. \tag{43}$$

From (32), it follows that

$$AM^2 = \frac{2\left(\dfrac{16}{25} + 1\right) - \dfrac{41 + \sqrt{1033 - 324\sqrt{3}}}{25}}{4} \quad \Leftrightarrow \quad AM^2 = \frac{41 - \sqrt{1033 - 324\sqrt{3}}}{100}$$

$$\Leftrightarrow \quad AM = \frac{\sqrt{41 - \sqrt{1033 - 324\sqrt{3}}}}{10}$$

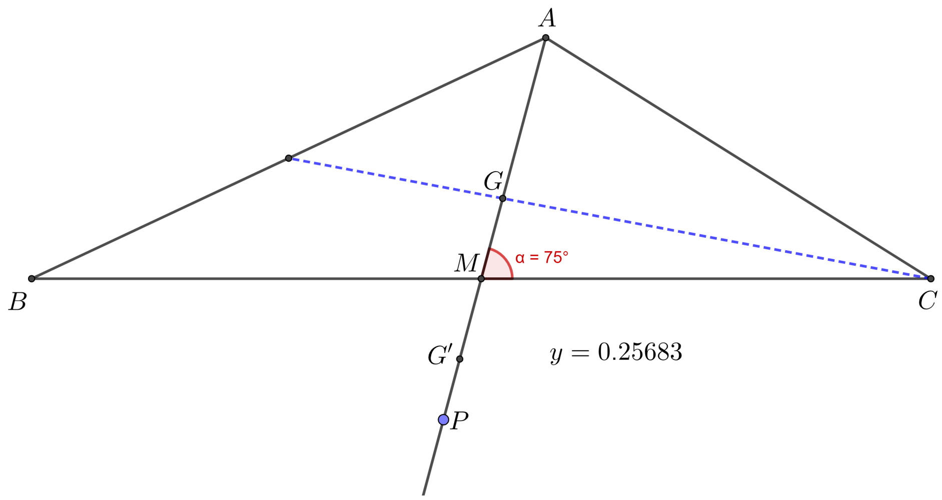
$$\Leftrightarrow \quad AM \approx 0.43907499019.$$

On using (33), we obtain

$$\begin{aligned}
2a \cdot AM \cdot \cos\alpha = c^2 - b^2 &\Leftrightarrow \sqrt{41^2 - 1033 + 324\sqrt{3}} \cdot \cos\alpha = 9\\
&\Leftrightarrow \sqrt{2^2 \cdot 9^2 \cdot \left(2 + \sqrt{3}\right)} \cdot \cos\alpha = 9\\
&\Leftrightarrow \cos\alpha = \frac{1}{2\sqrt{2+\sqrt{3}}}\\
&\Leftrightarrow \cos\alpha = \frac{\sqrt{2-\sqrt{3}}}{2}\\
&\Leftrightarrow \alpha \approx 1.308996939 \quad \text{radians}\\
&\Leftrightarrow \alpha \approx 75.00001 \quad \text{degrees.}
\end{aligned}$$

Next, we take advantage of the numerical values of the sides of the triangle ABC given in (43) to build it using Geogebra, according to the following picture.

Figure 6: Triangle ABC with $BC = \dfrac{\sqrt{41 + \sqrt{1033 - 324\sqrt{3}}}}{5}$, $AC = \dfrac{4}{5}$ and $AB = 1$.



Source: figure created by authors using Geogebra.

Above picture shows point P located at a position in which

$$y = PM \approx 0.25683.$$

Thus, one can use the program to verify the value of

$$\alpha = \angle AMC \approx 75.0 \text{ degrees},$$

and to measure dynamically the distance from P to M by moving point P over the straight line defined by A and M.

An interactive dynamic simulation of the triangle ABC given in Figure **??** can be visualized from the hyperlink < here>, moving point P.

From our reasoning, it is apparent that when P coincides with M, the midpoint of side BC, the distance from P to M equals zero and the harmonic mean reaches its minimum value which is given by

$$\begin{aligned}\frac{a^2}{4} &= \frac{41+\sqrt{1033-324\sqrt{3}}}{100}\\ &\approx 0.62721315299\\ &\approx 0.627.\end{aligned}$$

Also, it follows from (43) that

$$\frac{1}{2}-\cos^2\alpha=\frac{2}{4}-\frac{2-\sqrt{3}}{4}=\frac{\sqrt{3}}{4},$$

$$a^2=\frac{41+\sqrt{1033-324\sqrt{3}}}{25},$$

and

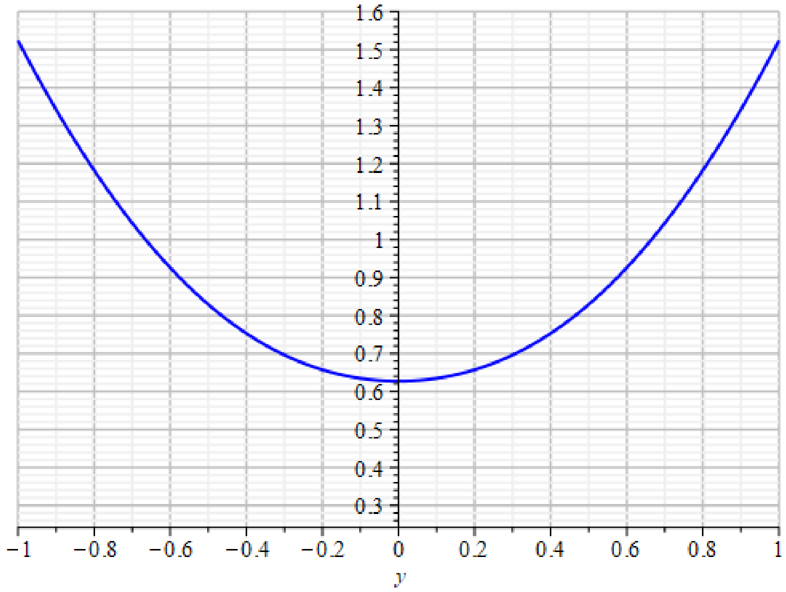
$$\begin{aligned}a^4 &= \left(\frac{41+\sqrt{1033-324\sqrt{3}}}{25}\right)\left(\frac{41+\sqrt{1033-324\sqrt{3}}}{25}\right)\\ &= \frac{41^2+\left(1033-324\sqrt{3}\right)+82\sqrt{1033-324\sqrt{3}}}{625}\\ &= \frac{2\left(1357-162\sqrt{3}+41\sqrt{1033-324\sqrt{3}}\right)}{625}.\end{aligned}$$

Hence, according to expression (8), it is clear that the harmonic mean as a function of $y=PM$ takes the form

$$\begin{aligned}f(y) &= \frac{2}{\dfrac{1}{PB^2}+\dfrac{1}{PC^2}}\\ &= \frac{4y^4+\dfrac{\sqrt{3}\left(41+\sqrt{1033-324\sqrt{3}}\right)}{25}y^2+\dfrac{1357-162\sqrt{3}+41\sqrt{1033-324\sqrt{3}}}{1250}}{4y^2+\dfrac{41+\sqrt{1033-324\sqrt{3}}}{25}}.\end{aligned}$$

The figure below illustrates the behaviour of this harmonic mean f for the numerical values of the sides of triangle ABC given in (43).

Figure 7: Graphic of the harmonic mean as function of $y = PM$.



Source: figure created by authors using MAPLE.

From the graph we also note that minimal value of the harmonic mean f occurs at $y = 0$ with $f(0) \approx 0.63$.

Conclusion

In this article, using a triangle ABC whose straight line AM is the median relative to the side BC, we have investigated the behaviour of the harmonic mean function of the squares of PB and PC for any point P on AM.

We have shown, amongst other algebraic results, that P is a relative extremum point of this harmonic mean, and centre of gravity of the triangle if and only if the sides of the triangle are roots of certain homogeneous polynomial of degree six.

We have also presented some numerical simulations to illustrate the study carried out.

The work developed here proves to be important by bringing additional properties to the triangle's centroid, namely, it provides a characterization of the triangles that have the centroid as an extremum of the aforementioned harmonic mean function. The subject has broad possibilities for further development, considering other cevians and other functions on them.

A possible limitation to the study would be the difficulty in algebraically finding the roots of the function to be studied, depending on the chosen cevian and the chosen function itself.

References

BIALOSTOCKI, A. and BIALOSTOCKI, D. The incenter and an excenter as solutions to an extremal problem. **Forum Geometricorum**, 11, 9–12, 2011. Available at: https://forumgeom.fau.edu/FG2011volume11/FG201102index.html (Accessed: 31 July 2023).

BIALOSTOCKI, A. and ELY, R. Points on a line that maximize and minimize the ratio of the distances to two given lines, **Forum Geometricorum**, 15, 177-178, 2015. Available at: https://forumgeom.fau.edu/FG2015volume15/FG201517.pdf (Accessed: 31 July 2023).

FREITAS, R. C. M., SANTOS, R. C., VELASCO, F. C. O Baricentro como ponto crítico de uma função. **Professor de Matemática Online** - PMO v.12, n.1, 2024 ISSN: 2319-023X. Available at: https://doi.org/10.21711/2319023x2024/pmo124 (Accessed: 11 November 2024).

HAJJA, M. Extremal properties of the incentre and the excentres of a triangle, **The Mathematical Gazette**, 2012, Vol. 96, No. 536, pp. 315-317, 2012.

HAJJA, M. One more note on the extremal properties of the incentre and the excentres of a triangle, **The Mathematical Gazette**, Vol. 101, No. 551, pp. 308-310, 2017.

SANTOS, R. C., FREITAS, R. C. M. (2024). O Baricentro como Ponto Crítico da função Média Geométrica entre duas determinadas distâncias em um triângulo qualquer. **Intermaths**, 5(1), 108-117, 2024. Available at: https://doi.org/10.22481/intermaths.v5i1.14238 (Accessed: 18 December 2024).