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On the properties of Lucas-balancing numbers by matrix method

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Abstract: *Balancing numbers n and balancers r are originally defined as the solution of the Diophantine equation $1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$. If n is a balancing number, then $8n^2+1$ is a perfect square. Further, If n is a balancing number then the positive square root of $8n^2+1$ is called a Lucas-balancing number. These numbers can be generated by the linear recurrences $B_{n+1}=6B_n-B_{n-1}$ and $C_{n+1}=6C_n-C_{n-1}$ where B_n and C_n are respectively denoted by the n^{th} balancing number and n^{th} Lucas-balancing number. There is another way to generate balancing and Lucas-balancing numbers using powers of matrices*

$$Q_B=\begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } Q_C=\begin{pmatrix} 17 & -3 \\ 3 & -1 \end{pmatrix}.$$

The matrix representation, indeed gives many known and new formulas for balancing and Lucas-balancing numbers. In this paper, using matrix algebra we obtain several interesting results on Lucas-balancing numbers.

Keywords: Balancing numbers, Lucas-balancing numbers, Balancing matrix, Lucas-balancing matrix.

Introduction

Behera and Panda (1999) recently introduced a number sequence called balancing numbers defined in the following way: A positive integer n is called a balancing number with balancer r, if it is the solution of the Diophantine equation $1+2+\ldots+(n-1)=(n+1)+(n+2)+\ldots+(n+r)$.

They also proved that the recurrence relation for balancing numbers is

$$B_{n+1}=6B_n-B_{n-1}, \quad n\geq 2, \tag{1}$$

where B_n is the n^{th} balancing number with $B_1=1$ and $B_2=6$.

It is well known that n is a balancing number if and only if n^2 is a triangular number, that is $8n^2+1$ is a perfect square (BEHERA; PANDA, 1999). Since $0=8\cdot 0^2+1$ is a perfect square, $B_0=0$ is also accepted as balancing number. In Panda (2009), Lucas-balancing numbers C_n are defined by $C_n=\sqrt{8B_n^2+1}$ where B_n is the n^{th} balancing number. The recurrence relation for Lucas-balancing numbers is same as that of balancing numbers, that is

$$C_{n+1}=6C_n-C_{n-1}, \quad n\geq 2, \tag{2}$$

with $C_1=3$ and $C_2=17$. Liptai (2004), showed that the only balancing number in the sequence of Fibonacci numbers is 1. In Ray (2012) and Ray (2013), Ray obtained nice product formulas for both balancing and Lucas-balancing numbers. Panda and Ray (2011) linked balancing numbers with Pell and associated Pell numbers. There is another way to generate balancing numbers using powers of matrices

$$Q_B=\begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix},$$

known as balancing matrix introduced in Ray (2012). Many interesting properties of balancing numbers using matrix method are also studied in Ray (2012).

The matrix representation indeed gives many known and new formulas for balancing numbers. In this paper, using matrix algebra we obtain several interesting results on Lucas-balancing numbers.

As in Ray (2012), the balancing matrix Q_B is in the form

$$Q_B = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix},$$

whose entries are first three balancing numbers. Also in Ray (2012), it has been shown that for integer $n \geq 1$, the n^{th} power of Q_B i.e. Q_B^n is given by

$$Q_B^n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix}. \tag{3}$$

The identity (3) provides an alternate proof of the Cassini's formula for balancing numbers (PANDA; RAY, 2011),

$$B_n^2 - B_{n+1}B_{n-1} = 1.$$

Since the identity $Q_B^{m+n} = Q_B^m Q_B^n$ is true for all integers m, n with $n \geq 1$, the following identities are straightforward:

$$B_{m+n+1} = B_{m+1}B_{n+1} - B_m B_n \text{ and } B_{m+n} = B_{m+1}B_n - B_m B_{n-1}.$$

These two results are basically similar, but could be applied to derive new identities for balancing numbers such as the following:

$$\begin{aligned} C_{m+n} &= B_{m+1}C_n - B_m C_{n-1}, \\ B_{m+n} &= B_m C_n + B_n C_m, \\ C_{m+n} &= C_m C_n + 8B_m B_n, \end{aligned}$$

where B_n and C_n are n^{th} balancing number and n^{th} Lucas-balancing number respectively. The following properties of balancing and Lucas-balancing numbers are given in (PANDA; RAY, 2011).

$$B_{n+1} - B_{n-1} = 2C_n \text{ and } C_{n+1} - C_{n-1} = 16B_n.$$

In this study, we define Lucas-balancing matrix Q_C by

$$Q_C = \begin{pmatrix} 17 & -3 \\ 3 & -1 \end{pmatrix}, \tag{4}$$

whose entries are first three Lucas-balancing numbers 1, 3 and 17. It is easy to verify that

$$\begin{pmatrix} C_{n+1} \\ C_n \end{pmatrix} = Q_C \begin{pmatrix} B_n \\ B_{n-1} \end{pmatrix} \text{ and } 8\begin{pmatrix} B_{n+1} \\ B_n \end{pmatrix} = \mathrm{Q_C} \begin{pmatrix} C_n \\ C_{n-1} \end{pmatrix}.$$

Matrix representation of Lucas-balancing numbers

In this section, we present a new matrix representation of both balancing and Lucas-balancing numbers. We obtain Cassini's formulas and some interesting properties of these numbers by a similar method as applied to the balancing Q_B matrix in Ray (2012). Our aim is not to compute the power of matrices, rather we find different relations between matrices containing balancing and Lucas-balancing numbers.

The following theorem establishes an interesting relation between balancing Q_B matrix and Lucas-balancing Q_C matrix.

Theorem 1: Let Q_C be the Lucas-balancing matrix as in (4). Then for integers $n \geq 1$,

$$Q_C^n = \begin{cases} 8^{\frac{n}{2}} \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix} & \text{for n even;} \\ 8^{\frac{n-1}{2}} \begin{pmatrix} C_{n+1} & -C_n \\ C_n & -C_{n-1} \end{pmatrix}, & \text{for n odd;} \end{cases} \tag{5}$$

where B_n and C_n are n^{th} balancing and Lucas-balancing numbers respectively.

Proof.: The proof proceeds by induction on n. First we consider odd n. The relation (5) is indeed true for $n = 1$, because $Q_C = \begin{pmatrix} C_2 & -C_1 \\ C_1 & C_0 \end{pmatrix}$, where $C_2 = 17, C_1 = 3, C_0 = 1$. We assume that suppose it is true for $n = k$, where k is an odd number. That is

$$Q_C^k = 8^{\frac{k-1}{2}} \begin{pmatrix} C_{k+1} & -C_k \\ C_k & -C_{k-1} \end{pmatrix}.$$

By using properties of Lucas-balancing numbers and induction hypothesis, we can write

$$Q_C^{k+2} = Q_C^k Q_C^2 = 8^{\frac{k+1}{2}} \begin{pmatrix} C_{k+3} & -C_{k+2} \\ C_{k+2} & -C_{k+1} \end{pmatrix},$$

as desired. Secondly, we consider even n. It is clear that the relation (5) is true for $n = 2$. We assume that suppose it is true for $n = k$, where k is an even number. That is

$$Q_B^k = 8^{\frac{k}{2}} \begin{pmatrix} B_{k+1} & -B_k \\ B_k & -B_{k-1} \end{pmatrix}.$$

By using properties of balancing numbers and induction hypothesis, we can write

$$Q_B^{k+2} = Q_B^k Q_B^2 = 8^{\frac{k+2}{2}} \begin{pmatrix} B_{k+3} & -B_{k+2} \\ B_{k+2} & -B_{k+1} \end{pmatrix},$$

as desired. Hence the relation (5) holds for all n.

□

Theorem 2: Let Q_C^n be as in (5). Then for all integers $n \geq 1$, the following identities are valid:

$$\begin{aligned} &i. \ \ \det(Q_C^n) = (-1)^n 8^n \\ &ii. \ \ B_n^2 - B_{n+1}B_{n-1} = 1 \\ &iii. \ \ C_{n+1}C_{n-1} - C_n^2 = 8 \end{aligned}$$

Proof.: To establish (i), we use induction on n. Clearly $\det(Q_C) = -8$. Assuming $\det(Q_C^k) = (-1)^k 8^k$, and by the multiplicative property of the determinant, we obtain

$$\det(Q_C^{k+1}) = \det(Q_C^k)\det(Q_C) = (-1)^k 8^k(-8) = (-1)^{k+1}8^{k+1},$$

which shows (i) for all $n \geq 1$. The identities (ii) and (iii) can be easily seen by using (5) and (i) for even and odd values of n, respectively. Second and third identities are indeed the Cassini formulas for balancing and Lucas-balancing numbers.

□

The Binet's formulas for balancing and Lucas-balancing numbers were established in Behera and Panda (1999). The following theorem is another approach to establish Binet's formulas with the help of linear algebra.

Theorem 3: For all integers n, the Binet's formula for balancing and Lucas-balancing numbers are respectively given by

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}}, \quad C_n = \frac{\lambda_1^n + \lambda_2^n}{2},$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.

Proof.: Let Q_C be the matrix as in (4). By solving the characteristic equation $\alpha^2 - 16\alpha - 8 = 0$ of Q_C, we obtain the eigenvalues and their corresponding eigenvectors as

$$\alpha_1 = \sqrt{8}\lambda_1,\ \alpha_2 = -\sqrt{8}\lambda_2 \ \text{ and } \ \mathrm{v}_1 = (1,\ \lambda_2),\ \mathrm{v}_2 = (1,\ \lambda_1),$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$. We now consider the matrices $A = (v_1^T,\ v_2^T) = \begin{pmatrix} 1 & 1 \\ \lambda_2 & \lambda_1 \end{pmatrix}$ and $B = dig(\lambda_1,\ \lambda_2) = \begin{pmatrix} \sqrt{8}\lambda_1 & 0 \\ 0 & -\sqrt{8}\lambda_2 \end{pmatrix}$ to diagonalize the matrix Q_C by $B = A^{-1}Q_C A$. By using the properties of similar matrices, for any integer n, we can write $B^n = A^{-1}Q_C^n A$. Furthermore, $Q_C^n = AB^n A^{-1}$. Therefore, taking the n^{th} power of diagonal matrix B, we obtain

$$Q_C^n = \frac{8^{n-1}}{2}\begin{pmatrix} \lambda_1^{n+1} - (-1)^n\lambda_2^{n+1} & -\lambda_1^n + (-1)^n\lambda_2^n \\ \lambda_1^n - (-1)^n\lambda_2^n & -\lambda_1^{n-1} + (-1)^n\lambda_2^{n-1} \end{pmatrix}.$$

The proof directly follows from equation (5) for n as even and odd respectively.

□

Theorem 4: For all integers m and n, the following identities are valid:

$$\begin{aligned}
&i).\ \ 8B_{m+n} = C_nC_{m+1} - C_{n-1}C_m \\
&ii).\ \ B_{m+n} = B_nB_{m+1} - B_{n-1}B_m \\
&iii).\ \ C_{m+n} = B_{n+1}C_m - B_nC_{m-1} \\
&iv).\ \ 8B_{m-n} = C_{m+1}C_n - C_mC_{n+1} \\
&v).\ \ B_{m-n} = B_mB_{n+1} - B_{m+1}B_n \\
&vi).\ \ C_{m-n} = B_{m+1}C_n - B_mC_{n+1}
\end{aligned}$$

Proof.: By (5), we can write Q_C^{m+n} as

$$Q_C^{m+n} = \begin{cases} 8^{\frac{m+n}{2}}\begin{pmatrix} B_{m+n+1} & -B_{m+n} \\ B_{m+n} & -B_{m+n-1} \end{pmatrix} & \text{for m+n even;} \\ 8^{\frac{m+n-1}{2}}\begin{pmatrix} C_{m+n+1} & -C_{m+n} \\ C_{m+n} & -C_{m+n-1} \end{pmatrix} & \text{for m+n odd.} \end{cases} \tag{6}$$

For the case of odd m and n,

$$Q_C^m Q_C^n = 8^{\frac{m+n}{2}-1}\begin{pmatrix} C_{m+1}C_{n+1} - C_mC_n & C_mC_{n-1} - C_{m+1}C_n \\ C_mC_{n+1} - C_{m-1}C_n & C_{m-1}C_{n-1} - C_mC_n \end{pmatrix}. \tag{7}$$

Comparing the first row second column entries from both the matrices (6) and (7), we obtain

$$8B_{m+n} = C_nC_{m+1} - C_{n-1}C_m,$$

while the second row first column entries gives

$$8B_{m+n} = C_m C_{n+1} - C_{m-1} C_n.$$

For the case of even m and n,

$$Q_C^m Q_C^n = 8^{\frac{m+n}{2}} \begin{pmatrix} B_{m+1}B_{n+1} - B_m B_n & B_m B_{n-1} - B_{m+1} B_n \\ B_m B_{n+1} - B_{m-1} B_n & B_{m-1} B_{n-1} - B_m B_n \end{pmatrix}. \tag{8}$$

Comparing the first row second column entries from both the matrices (6) and (8), we obtain

$$B_{m+n} = B_n B_{m+1} - B_{n-1} B_m,$$

while the second row first column entries gives

$$B_{m+n} = B_m B_{n+1} - B_{m-1} B_n.$$

For the cases odd m and even n or even m and odd n,

$$Q_C^m Q_C^n = 8^{\frac{m+n-1}{2}} \begin{pmatrix} B_{m+1}C_{n+1} - B_m C_n & B_m C_{n-1} - B_{m+1} C_n \\ B_m C_{n+1} - B_{m-1} C_n & B_{m-1} C_{n-1} - B_m C_n \end{pmatrix}. \tag{9}$$

Comparing the first row second column entries from both the matrices (6) and (9), we obtain

$$C_{m+n} = B_m C_{n-1} - B_{m+1} C_n,$$

while the second row first column entries gives

$$B_{m+n} = B_m C_{n+1} - B_{m-1} C_n.$$

The inverse of the matrix Q_C^n is given by

$$Q_C^{-n} = \begin{cases} \frac{1}{8^{\frac{n}{2}}} \begin{pmatrix} -B_{n-1} & B_n \\ -B_n & B_{n+1} \end{pmatrix} & \text{for n even;} \\ \frac{-1}{8^{\frac{n+1}{2}}} \begin{pmatrix} -C_{n-1} & C_n \\ -C_n & C_{n+1} \end{pmatrix}, & \text{for n odd.} \end{cases} \tag{10}$$

In a similar manner, by computing the equality $Q_{m-n} = Q_m Q_{-n}$, the desired results are obtained. Indeed, for the case of odd m and n,

$$8B_{m-n} = C_{m+1} C_n - C_m C_{n+1}.$$

For the case of even m and n,

$$B_{m-n} = B_m B_{n+1} - B_{m+1} B_n.$$

Finally, for the case of odd m and even n or even m and odd n,

$$C_{m-n} = B_m C_{n-1} - B_{m-1} C_n.$$

This completes the proof of the theorem.

$\square$

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