ISSN: 2317-0840

Gaussian bi-periodic Fibonacci and Gaussian bi-periodic Lucas Sequences

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Abstract: *In this study, we bring into light of the gaussian bi-periodic Fibonacci and gaussian bi-periodic Lucas sequences. The Binet formula as well as the generating function for these sequences are given. The convergence property of the consecutive terms of this sequence is examined after the well-known Cassini, Catalan and the D'ocagne identities as well as some related summation formulas are also given.*

Keywords: Bi-periodic Fibonacci sequence; Bi-periodic Lucas sequence; Generating function; Binet formula.

Introductıon

Due to the numerous applications of integer sequences such as Fibonacci, Lucas, Jacobsthal, Jacobsthal-Lucas, Pell etc in many fields of science and art, there have been many generalizations on them over the last century. You can see some of these different generalizations in all our references. Fibonacci and Lucas sequences are defined by the recurrence relations $f_n = f_{n-1}+f_{n-2}$ with the initial conditions $f_0 = 0$, $f_1 = 1$. The classical Lucas sequence is defined as $l_n = l_{n-1}+l_{n-2}$ with the initial conditions $l_0 = 2$, $l_1 = 1$ respectively. For any natural number n and any nonzero real numbers a and b, the bi-periodic Fibonacci sequence was defined recursively by Edson and Yayenie (2009), Yayenie (2011) and Yayenie (2012) as

$$q_n = \begin{cases} aq_{n-1}+q_{n-2} & if\ n\ is\ even \\ bq_{n-1}+q_{n-2} & if\ n\ is\ odd \end{cases} \quad (n \geq 2) \tag{1}$$

with the initial conditions $q_0 = 0,\ q_1 = 1$. Bilgici (2014) defined the bi-periodic Lucas sequence by the following recursion relation

$$L_n = \begin{cases} bL_{n-1}+L_{n-2} & if\ n\ is\ even \\ aL_{n-1}+L_{n-2} & if\ n\ is\ odd \end{cases} \quad (n \geq 2) \tag{2}$$

with the initial conditions $L_0 = 2,\ L_1 = a$. He also found some interesting identities between the above two sequences. From (1) and (2), the nonlinear quadratic equation for the bi-periodic Fibonacci and the bi-periodic Lucas sequences is

$$x^2 - abx - ab = 0$$

with roots

$$\alpha = \frac{ab+\sqrt{a^2b^2+4ab}}{2} \quad \text{and} \quad \beta = \frac{ab-\sqrt{a^2b^2+4ab}}{2}. \tag{3}$$

α and β defined by (3) satisfy the following properties:

1. $(\alpha+1)(\beta+1)=1$
2. $\alpha+\beta=ab,\ \ \alpha\beta=-ab$
3. $\alpha+1=\frac{\alpha^2}{ab},\ \beta+1=\frac{\beta^2}{ab}$
4. $-\beta(\alpha+1)=\alpha,\ -\alpha(\beta+1)=\beta.$

For every n belonging to the set of natural numbers, the Binet formula for the bi-periodic Fibonacci sequence is

$$q_n=\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) \tag{4}$$

and the Binet formula for the bi-periodic Lucas sequence is given by

$$L_n=\frac{a^{\xi(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}(\alpha^n+\beta^n), \tag{5}$$

where $\lfloor a \rfloor$ is the floor function of a and $\xi(n)=n-2\left\lfloor\frac{n}{2}\right\rfloor$ is the parity function (EDSON; YAYENİE, 2009; YAYENİE, 2011; YAYENİE, 2012; BİLGİCİ, 2014).

Uygun and Owusu (2016, 2019) gave some basic properties of the bi-periodic Jacobsthal and bi-periodic Jacobsthal Lucas sequences. Uygun and Karatas (2019, 2020) introduced the bi-periodic Pell and Pell Lucas sequences. In Younseok (2019), Choo examined some identities of generalized bi-periodic Fibonacci sequences. Coskun and Taskara (2018) defined the bi-periodic Fibonacci and Lucas matrix sequences. Fibonacci and Lucas sequences have recieved so much attention over the years.

The authors have made many generalizations for them. Jordan (1965), Harman (1981) and Pethe and Horadam (1986) carried out Fibonacci and Lucas sequences to complex plane. Aşcı and Gurel (2013) studied gaussian Jacobsthal and gaussian Jacobsthal Lucas numbers. Gaussian Pell and Pell-Lucas numbers were introduced by Halici and Öz (2016). In this note we made a new generalization for the Fibonacci and Lucas sequences which we shall call them the gaussian bi-periodic Fibonacci and the gaussian bi-periodic Lucas sequences. We will then proceed to find their generating functions as well as the Binet formulas for them. The convergence properties of the consecutive terms of these sequences will be examined after which Cassini, Catalan and D'ocagne identities as well as some related formulas and properties will be given.

Basic properties of Gaussian bi-Periodic Fibonacci and Gaussian bi-Periodic Lucas Sequences

Definition 1 For any two non-zero real numbers a and b, a new generalization for the Fibonacci sequence $\{\mathfrak{F}_m\}_{m=0}^{\infty}$, which called the gaussian bi-periodic Fibonacci sequence is defined recursively by

$$\mathfrak{F}_n=\begin{cases} a\mathfrak{F}_{n-1}+\mathfrak{F}_{n-2} & if\ n\ is\ even \\ b\mathfrak{F}_{n-1}+\mathfrak{F}_{n-2} & if\ n\ is\ odd \end{cases} \quad (n\geq 2) \tag{6}$$

with $\mathfrak{F}_0 = ai,\ \mathfrak{F}_1 = 1.$

For any two non-zero real numbers a and b, a new generalization for the Lucas sequence $\{\mathfrak{L}_m\}_{m=0}^{\infty}$, which called the gaussian bi-periodic Lucas sequence is defined recursively by

$$\mathfrak{L}_n = \begin{cases} b\mathfrak{L}_{n-1} + \mathfrak{L}_{n-2} & if\ n\ is\ even \\ a\mathfrak{L}_{n-1} + \mathfrak{L}_{n-2} & if\ n\ is\ odd \end{cases} \quad (n \geq 2) \tag{7}$$

with $\mathfrak{L}_0 = 2 - abi,\ \mathfrak{L}_1 = (1 + 2i)a.$

The first five elements of the gaussian bi-periodic Fibonacci sequence are

$$\mathfrak{F}_0 = ai, \quad \mathfrak{F}_1 = 1, \quad \mathfrak{F}_2 = a + ai, \quad \mathfrak{F}_3 = ab + 1 + abi, \quad \mathfrak{F}_4 = (a^2b + 2a + a^2b + a)i.$$

The first five elements of the gaussian bi-periodic Lucas sequence are

$$\mathfrak{L}_0 = 2 - abi, \quad \mathfrak{L}_1 = (1 + 2i)a, \quad \mathfrak{L}_2 = ab + 2 + abi, \quad \mathfrak{L}_3 = a^2b + 3a + (a^2b + 2a)i,$$
$$\mathfrak{L}_4 = a^2b^2 + 4ab + 2 + (a^2b^2 + 3ab)i.$$

The Binet formula for the gaussian bi-periodic Fibonacci sequence as follows

$$\mathfrak{F}_n = \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{(1 - i\,\beta)\alpha^n - (1 - i\alpha)\beta^n}{\alpha - \beta} \right). \tag{8}$$

The Binet formula for the gaussian bi-periodic Lucas sequence is given by

$$\mathfrak{L}_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} ((1 - i\,\beta)\alpha^n + (1 - i\alpha)\beta^n). \tag{9}$$

The relation between the bi-periodic Fibonacci and the gaussian bi-periodic Fibonacci and the relation between the bi-periodic Lucas sequence and the gaussian bi-periodic Lucas sequence are summarized as

$$\mathfrak{F}_n = \begin{cases} q_n + aiq_{n-1} & if\ n\ is\ even \\ q_n + biq_{n-1} & if\ n\ is\ odd \end{cases}$$
$$\mathfrak{L}_n = \begin{cases} L_n + aiL_{n-1} & if\ n\ is\ even \\ L_n + biL_{n-1} & if\ n\ is\ odd \end{cases}$$

Lemma 2 The gaussian bi-periodic Fibonacci and Lucas sequences satisfy the following properties:

$$\mathfrak{F}_{2n} = (ab + 2)\mathfrak{F}_{2n-2} - \mathfrak{F}_{2n-4},$$
$$\mathfrak{F}_{2n-1} = (ab + 2)\mathfrak{F}_{2n-3} - \mathfrak{F}_{2n-5},$$
$$\mathfrak{L}_{2n} = (ab + 2)\mathfrak{L}_{2n-2} - \mathfrak{L}_{2n-4},$$
$$\mathfrak{L}_{2n-1} = (ab + 2)\mathfrak{L}_{2n-3} - \mathfrak{L}_{2n-5}$$

Proof: By the recurrence relations of gaussian bi-periodic Fibonacci and Lucas sequences, we get

$$\begin{aligned} \mathfrak{F}_{2n} &= a\mathfrak{F}_{2n-1} + \mathfrak{F}_{2n-2} \\ &= a(\,b\mathfrak{F}_{2n-2} + \mathfrak{F}_{2n-3}) + \mathfrak{F}_{2n-2} \\ &= (ab + 1)\mathfrak{F}_{2n-2} + (\mathfrak{F}_{2n-2} - \mathfrak{F}_{2n-4}) \end{aligned}$$

$$= (ab+2)\mathfrak{F}_{2n-2} - \mathfrak{F}_{2n-4}$$

The other proofs are made similarly.

Theorem 3: The generating function for the gaussian bi-periodic Fibonacci sequence is demonstrated by

$$F(x) = \frac{ai + x + (a + ai - a^2bi - 2ai)x^2 + (abi - 1)x^3}{1 - (ab+2)x^2 + x^4}.$$

Proof: The generating function is given as

$$F(x) = \sum_{\mathrm{m}=0}^{\infty} \mathfrak{F}_m x^m = F_0(x) + F_1(x) = \sum_{\mathrm{m}=0}^{\infty} \mathfrak{F}_{2m} x^{2m} + \sum_{\mathrm{i}=0}^{\infty} \mathfrak{F}_{2m+1} x^{2m+1}.$$

We simplify the even part of the above series as follows

$$F_0(x) = \sum_{\mathrm{m}=0}^{\infty} \mathfrak{F}_{2m} x^{2m} = ai + a(1+i)x^2 + \sum_{\mathrm{m}=2}^{\infty} \mathfrak{F}_{2m} x^{2m}.$$

By multiplying through by $(ab+2)x^2$ and x^4 respectively, we have

$$(ab+2)x^2 F_0(x) = ai(ab+2)x^2 + (ab+2)\sum_{\mathrm{m}=2}^{\infty} \mathfrak{F}_{2m-2} x^{2m},$$

and

$$x^4 F_0(x) = \sum_{\mathrm{m}=2}^{\infty} \mathfrak{F}_{2m-4} x^{2m}.$$

By using Lemma 2, it is obtained that

$$F_0(x) = \frac{ai + (a + ai - a^2bi - 2ai)x^2}{1 - (ab+2)x^2 + x^4}.$$

Similarly, the odd part of the above series is simplified as follows

$$F_1(x) = \sum_{\mathrm{m}=0}^{\infty} \mathfrak{F}_{2m+1} x^{2m+1} = x + (ab + 1 + abi)x^3 + \sum_{\mathrm{m}=2}^{\infty} \mathfrak{F}_{2m+1} x^{2m+1}.$$

By multiplying through by $(ab+2)x^2$ and x^4 respectively, we have

$$(ab+2)x^2 F_1(x) = (ab+2)x^3 + (ab+2)\sum_{\mathrm{m}=2}^{\infty} \mathfrak{F}_{2m-1} x^{2m+1},$$

and

$$x^4 F_1(x) = \sum_{\mathrm{m}=2}^{\infty} \mathfrak{F}_{2m-3} x^{2m+1}.$$

By using Lemma 2, it is obtained that

$$F_1(x) = \frac{x + (abi - 1)x^3}{1 - (ab + 2)x^2 + x^4}.$$

By combining the two results, we have

$$F(x) = F_0(x) + F_1(x)$$
$$= \frac{ai + x + (a + ai - a^2bi - 2ai)x^2 + (abi - 1)x^3}{1 - (ab + 2)x^2 + x^4}.$$

Theorem 4: The generating function for the gaussian bi-periodic Lucas sequence is denoted by

$$L(x) = \frac{2 - abi + (1 + 2i)ax + (-ab - 2 + (a^2b^2 + 3ab)i)x^2 + (a - (a^2b + 2a))x^3}{1 - (ab + 2)x^2 + x^4}$$

Proof: The proof is similar with the proof of Theorem 3.

Theorem 5: The limits of every two consecutive terms of the gaussian bi-periodic Fibonacci-Lucas sequence are generalized as

$$\lim_{n\to\infty} \frac{\mathfrak{F}_{2n+1}}{\mathfrak{F}_{2n}} = \frac{\alpha}{a}, \qquad \lim_{n\to\infty} \frac{\mathfrak{F}_{2n}}{\mathfrak{F}_{2n-1}} = \frac{\alpha}{b}.$$
$$\lim_{n\to\infty} \frac{\mathfrak{L}_{2n+1}}{\mathfrak{L}_{2n}} = \frac{\alpha}{b}, \qquad \lim_{n\to\infty} \frac{\mathfrak{L}_{2n}}{\mathfrak{L}_{2n-1}} = \frac{\alpha}{a}.$$

Proof: Taking into account that $|\beta| < \alpha$ and $\lim_{n\to\infty} \left(\frac{\beta}{\alpha}\right)^n = 0,$ we have

$$\lim_{n\to\infty} \frac{\mathfrak{L}_{2n+1}}{\mathfrak{L}_{2n}} = \lim_{n\to\infty} \frac{\frac{a}{(ab)^{\left\lfloor\frac{2n+2}{2}\right\rfloor}}((1 - i\,\beta)\alpha^{2n+1} + (1 - i\alpha)\beta^{2n+1})}{\frac{1}{(ab)^{\left\lfloor\frac{2n+1}{2}\right\rfloor}}((1 - i\,\beta)\alpha^{2n} + (1 - i\alpha)\beta^{2n})}$$
$$= \lim_{n\to\infty} \frac{\frac{a}{(ab)^{\left\lfloor\frac{2n+2}{2}\right\rfloor}}\left((1 - i\,\beta) + (1 - i\alpha)\left(\frac{\beta}{\alpha}\right)^{2n+1}\right)}{\frac{1}{(ab)^{\left\lfloor\frac{2n+1}{2}\right\rfloor}}\left(\frac{(1 - i\,\beta)}{\alpha} + (1 - i\alpha)\left(\frac{\beta}{\alpha}\right)^{2n+1}\frac{1}{\beta}\right)} = \frac{\alpha}{b}.$$

The other proofs can be made similarly. From this theorem we can conclude that the gaussian bi-periodic Fibonacci and the gaussian bi-periodic Lucas sequences do not converge.

Theorem 6: (Binomial Sums): Let n be any nonnegative integer, then we have

$$\sum_{k=0}^{n} \binom{n}{k} a^{\xi(k)} (ab)^{\left\lfloor\frac{k}{2}\right\rfloor} \mathfrak{F}_k = \mathfrak{F}_{2n},$$
$$\sum_{k=0}^{n} \binom{n}{k} a^{\xi(k+1)} (ab)^{\left\lfloor\frac{k+1}{2}\right\rfloor} \mathfrak{F}_{k+1} = a\mathfrak{F}_{2n+1},$$

and

$$\sum_{k=0}^{n}\binom{n}{k}a^{\xi(k+1)}(ab)^{\left\lfloor\frac{k+1}{2}\right\rfloor}\mathfrak{L}_k = a\mathfrak{L}_{2n},$$
$$\sum_{k=0}^{n}\binom{n}{k}a^{\xi(k)}(ab)^{\left\lfloor\frac{k}{2}\right\rfloor}\mathfrak{L}_{k+1} = \mathfrak{L}_{2n+1}.$$

Proof:

$$\sum_{k=0}^{n}\binom{n}{k}a^{\xi(k)}(ab)^{\left\lfloor\frac{k}{2}\right\rfloor}\mathfrak{F}_k = \sum_{k=0}^{n}\binom{n}{k}a\left[\left(\frac{(1-i\,\beta)\alpha^n-(1-i\alpha)\beta^n}{\alpha-\beta}\right)\right]$$
$$= \frac{a}{\alpha-\beta}\left[(1-i\,\beta)\sum_{k=0}^{n}\binom{n}{k}\alpha^k-(1-i\alpha)\sum_{k=0}^{n}\binom{n}{k}\beta^k\right]$$
$$= \frac{a}{\alpha-\beta}[(1-i\,\beta)(\alpha+1)^n-(1-i\alpha)(\beta+1)^n]$$
$$= \frac{a}{\alpha-\beta}\left[(1-i\,\beta)\left(\frac{\alpha^2}{ab}\right)^n-(1-i\alpha)\left(\frac{\beta^2}{ab}\right)^n\right]$$
$$= \frac{a^{1-\xi(2n)}}{(ab)^{\left\lfloor\frac{2n}{2}\right\rfloor}}\left[\frac{(1-i\,\beta)\alpha^{2n}-(1-i\alpha)\beta^{2n}}{\alpha-\beta}\right]$$
$$= \mathfrak{F}_{2n}.$$

Theorem 7: (Catalan Identity) For all integers n and r, with $n \geq r$ the Catalan's identity for the gaussian bi-periodic Fibonacci and the gaussian bi-periodic Lucas sequences are given by

$$\left(\frac{a}{b}\right)^{\xi(n+r)}\mathfrak{F}_{n-r}\mathfrak{F}_{n+r}-\left(\frac{a}{b}\right)^{\xi(n)}\mathfrak{F}_n^2 = -(-1)^{n-r}(1-i\alpha)(1-i\,\beta)\left(\frac{a}{b}\right)^{\xi(r)}q_r^2$$

$$\left(\frac{b}{a}\right)^{\xi(n+r)}\mathfrak{L}_{n-r}\mathfrak{L}_{n+r}-\left(\frac{b}{a}\right)^{\xi(n)}\mathfrak{L}_n^2 = (-1)^{n-r}(1-i\alpha)(1-i\,\beta)(ab+2)\left(\frac{b}{a}\right)^{\xi(r+1)}q_r^2.$$

Proof: For the proof we use the following properties

$$\xi(n+r)+\left\lfloor\frac{n-r}{2}\right\rfloor+\left\lfloor\frac{n+r}{2}\right\rfloor = n,$$
$$\xi(n+r)-\left\lfloor\frac{n-r+1}{2}\right\rfloor-\left\lfloor\frac{n+r+1}{2}\right\rfloor = -n,$$
$$\xi(n)+2\left\lfloor\frac{n}{2}\right\rfloor = n,\quad \xi(n)-2\left\lfloor\frac{n+1}{2}\right\rfloor = -n.$$

By using Binet's formula, it is obtained that

$$b\left(\frac{a}{b}\right)^{\xi(n+r)}\mathfrak{F}_{n-r}\mathfrak{F}_{n+r} = b\left(\frac{a}{b}\right)^{\xi(n+r)}\left(\frac{a^{1-\xi(n-r)}}{(ab)^{\left\lfloor\frac{n-r}{2}\right\rfloor}}\right)\left(\frac{a^{1-\xi(n+r)}}{(ab)^{\left\lfloor\frac{n+r}{2}\right\rfloor}}\right)$$
$$\cdot\left(\frac{(1-i\,\beta)\alpha^{n-r}-(1-i\alpha)\beta^{n-r}}{\alpha-\beta}\right)\left(\frac{(1-i\,\beta)\alpha^{n+r}-(1-i\alpha)\beta^{n+r}}{\alpha-\beta}\right)$$

$$= \frac{a^{2-\xi(n+r)}b^{1-\xi(n+r)}}{(ab)^{n-\xi(n-r)}}\left(\frac{(1-i\,\beta)\alpha^{n-r}-(1-i\alpha)\beta^{n-r}}{\alpha-\beta}\right)\left(\frac{(1-i\,\beta)\alpha^{n+r}-(1-i\alpha)\beta^{n+r}}{\alpha-\beta}\right)$$

$$= \frac{a}{(ab)^{n-1}}\left(\frac{(1-i\,\beta)^2\alpha^{2n} - (1-i\alpha)(1-i\,\beta)(\alpha\beta)^{n-r}(\alpha^{2r}+\beta^{2r}) + (1-i\,\alpha)^2\beta^{2n}}{(\alpha-\beta)^2}\right)$$

$$b\left(\frac{a}{b}\right)^{\xi(n)}\mathfrak{F}_n^2 =$$

$$= a^{\xi(n)}b^{1-\xi(n)}\left(\frac{a^{2-2\xi(n)}}{(ab)^{2\left\lfloor\frac{n}{2}\right\rfloor}}\right)\left(\frac{(1-i\,\beta)^2\alpha^{2n} - 2(1-i\alpha)(1-i\,\beta)(\alpha\beta)^n + (1-i\,\alpha)^2\beta^{2n}}{(\alpha-\beta)^2}\right)$$

$$= \frac{a}{(ab)^{n-1}}\left(\frac{(1-i\,\beta)^2\alpha^{2n} - 2(1-i\alpha)(1-i\,\beta)(\alpha\beta)^n + (1-i\,\alpha)^2\beta^{2n}}{(\alpha-\beta)^2}\right).$$

Then

$$b\left(\frac{a}{b}\right)^{\xi(n+r)}\mathfrak{F}_{n-r}\mathfrak{F}_{n+r} - b\left(\frac{a}{b}\right)^{\xi(n)}\mathfrak{F}_n^2$$

$$= \frac{a}{(ab)^{n-1}}\left(\frac{(1-i\alpha)(1-i\,\beta)[-(\alpha\beta)^{n-r}(\alpha^{2r}+\beta^{2r})] + 2\,(\alpha\beta)^n}{(\alpha-\beta)^2}\right)$$

$$= \frac{a(1-i\alpha)(1-i\,\beta)}{(ab)^{n-1}}(\alpha\beta)^{n-r}\left(\frac{-\alpha^{2r}-\beta^{2r} + 2\,(\alpha\beta)^r}{(\alpha-\beta)^2}\right)$$

$$= \frac{-a(1-i\alpha)(1-i\,\beta)}{(ab)^{n-1}}(\alpha\beta)^{n-r}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)^2$$

$$= \frac{-(-1)^{n-r}a(1-i\alpha)(1-i\,\beta)}{(ab)^{r-1}}\frac{(ab)^{2\left\lfloor\frac{r}{2}\right\rfloor}}{a^{2-2\xi(r)}}\mathrm{q}_r{}^2$$

$$= -(-1)^{n-r}(1-i\alpha)(1-i\,\beta)a^{-1+2\xi(r)+1-\xi(r)}b^{1-\xi(r)}\mathrm{q}_r{}^2$$

$$= -b(-1)^{n-r}(1-i\alpha)(1-i\,\beta)\left(\frac{a}{b}\right)^{\xi(r)}\mathrm{q}_r{}^2$$

which completes the proof.

Corollary 8 (Cassini Identity) For any number n belonging to the set of positive integers, we have

$$\left(\frac{a}{b}\right)^{\xi(n+1)}\mathfrak{F}_{n-1}\mathfrak{F}_{n+1} - \left(\frac{a}{b}\right)^{\xi(n)}\mathfrak{F}_n^2 = (-1)^n(1-i\alpha)(1-i\,\beta)\frac{a}{b},$$

$$\left(\frac{b}{a}\right)^{\xi(n+1)}\mathfrak{L}_{n-1}\mathfrak{L}_{n+1} - \left(\frac{b}{a}\right)^{\xi(n)}\mathfrak{L}_n^2 = (-1)^{n-1}(1-i\alpha)(1-i\,\beta)(ab+2).$$

Proof: This is a special case of the Catalan identity in which the value of r is 1. Therefore the Cassini' s property can easily be proven by a mere substitution of $r = 1$ into the Catalan identity.

Theorem 9 (D'ocagne's property) For any numbers m and n, belonging to the set of positive integers, with $m \geq n$, we have

$$a^{\xi(mn+m)}b^{\xi(mn+n)}\mathfrak{F}_{n+1}\mathfrak{F}_m - a^{\xi(mn+n)}b^{\xi(mn+m)}\mathfrak{F}_n\mathfrak{F}_{m+1} = (-1)^n a^{\xi(m-n)}(1-i\alpha)(1-i\,\beta)q_{m-n},$$

$$a^{\xi(mn+m)}b^{\xi(mn+n)}\mathfrak{L}_{m+1}\mathfrak{L}_n - a^{\xi(mn+n)}b^{\xi(mn+m)}\mathfrak{L}_m\mathfrak{L}_{n+1}$$
$$= (-1)^n a^{\xi(m-n)}(ab+2)(1-i\alpha)(1-i\,\beta)q_{m-n}.$$

Proof:

$$\omega = a^{\xi(mn+m)}b^{\xi(mn+n)}\mathfrak{F}_{n+1}\mathfrak{F}_m = a^{\xi(mn+m)}b^{\xi(mn+n)}\left(\frac{a^{1-\xi(m)}}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor}}\right)\left(\frac{a^{1-\xi(n+1)}}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\right)$$
$$.\left(\frac{(1-i\,\beta)\alpha^m-(1-i\alpha)\beta^m}{\alpha-\beta}\right)\left(\frac{(1-i\,\beta)\alpha^{n+1}-(1-i\alpha)\beta^{n+1}}{\alpha-\beta}\right)$$

$$=\frac{ab^{\xi(mn+n)}a^{1-\xi(m)-\xi(n+1)+\xi(mn+m)}}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor}(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}.\left(\frac{(1-i\,\beta)\alpha^m-(1-i\alpha)\beta^m}{\alpha-\beta}\right)\left(\frac{(1-i\,\beta)\alpha^{n+1}-(1-i\alpha)\beta^{n+1}}{\alpha-\beta}\right)$$

$$=\frac{ab^{\xi(mn+n)}a^{\xi(m-n)-\xi(mn+m)}}{(ab)^{\frac{m-n-\xi(m-n)}{2}+\xi(mn+n)+n}}.\left(\frac{(1-i\,\beta)\alpha^m-(1-i\alpha)\beta^m}{\alpha-\beta}\right)\left(\frac{(1-i\,\beta)\alpha^{n+1}-(1-i\alpha)\beta^{n+1}}{\alpha-\beta}\right)$$

$$=\frac{ab^{\xi(mn+n)}a^{\xi(mn+n)}}{(ab)^{\frac{m-n-\xi(m-n)}{2}+\xi(mn+n)+n}}.\left(\frac{(1-i\,\beta)\alpha^m-(1-i\alpha)\beta^m}{\alpha-\beta}\right)\left(\frac{(1-i\,\beta)\alpha^{n+1}-(1-i\alpha)\beta^{n+1}}{\alpha-\beta}\right)$$
$$=\frac{a(ab)^{-n}}{(ab)^{\frac{m-n-\xi(m-n)}{2}}}$$
$$.\left(\frac{(1-i\,\beta)^2\alpha^{m+n+1}+(1-i\,\alpha)^2\beta^{m+n+1}-(1-i\alpha)(1-i\,\beta)(\alpha\beta)^n(\beta\alpha^{m-n}+\alpha\,\beta^{m-n}}{(\alpha-\beta)^2}\right).$$

$$\varphi = a^{\xi(mn+n)}b^{\xi(mn+m)}\mathfrak{F}_n\mathfrak{F}_{m+1} = a^{\xi(mn+n)}b^{\xi(mn+m)}\left(\frac{a^{1-\xi(n)}}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}\right)\left(\frac{a^{1-\xi(m+1)}}{(ab)^{\left\lfloor\frac{m+1}{2}\right\rfloor}}\right)$$
$$.\left(\frac{(1-i\,\beta)\alpha^n-(1-i\alpha)\beta^n}{\alpha-\beta}\right)\left(\frac{(1-i\,\beta)\alpha^{m+1}-(1-i\alpha)\beta^{m+1}}{\alpha-\beta}\right)$$

$$=\frac{ab^{\xi(mn+m)}a^{1-\xi(m+1)-\xi(n)+\xi(mn+n)}}{(ab)^{\left\lfloor\frac{m+1}{2}\right\rfloor}(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}.\left(\frac{(1-i\,\beta)\alpha^n-(1-i\alpha)\beta^n}{\alpha-\beta}\right)\left(\frac{(1-i\,\beta)\alpha^{m+1}-(1-i\alpha)\beta^{m+1}}{\alpha-\beta}\right)$$

$$=\frac{ab^{\xi(mn+m)}a^{\xi(m-n)-\xi(mn+n)}}{(ab)^{\frac{m-n-\xi(m-n)}{2}+\xi(mn+m)+n}}.\left(\frac{(1-i\,\beta)\alpha^n-(1-i\alpha)\beta^n}{\alpha-\beta}\right)\left(\frac{(1-i\,\beta)\alpha^{m+1}-(1-i\alpha)\beta^{m+1}}{\alpha-\beta}\right)$$

$$=\frac{a(ab)^{-n}}{(ab)^{\frac{m-n-\xi(m-n)}{2}}}$$
$$.\left(\frac{(1-i\,\beta)^2\alpha^{m+n+1}+(1-i\,\alpha)^2\beta^{m+n+1}-(1-i\alpha)(1-i\,\beta)(\alpha\beta)^n(\alpha^{m-n+1}+\,\beta^{m-n+1}}{(\alpha-\beta)^2}\right).$$

$$\omega-\varphi = a^{\xi(mn+m)}b^{\xi(mn+n)}\mathfrak{F}_{n+1}\mathfrak{F}_m - a^{\xi(mn+n)}b^{\xi(mn+m)}\mathfrak{F}_n\mathfrak{F}_{m+1}$$

$$=\frac{a(ab)^{-n}}{(ab)^{\frac{m-n-\xi(m-n)}{2}}}\left(\frac{(1-i\alpha)(1-i\,\beta)(\alpha\beta)^n(-\beta\alpha^{m-n}-\alpha\,\beta^{m-n}+\alpha^{m-n+1}+\beta^{m-n+1}}{(\alpha-\beta)^2}\right)$$

$$= \frac{a(-1)^n}{(ab)^{\left\lfloor \frac{m-n}{2} \right\rfloor}} \left(\frac{(1-i\alpha)(1-i\,\beta)(\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta} \right)$$

$$= (-1)^n a^{\xi(m-n)}(1-i\alpha)(1-i\,\beta)q_{m-n}.$$

Theorem 10: The sum formulas for the gaussian bi-periodic Fibonacci and the gaussian bi-periodic Lucas sequences are given

$$\sum_{k=0}^{n-1} \mathfrak{F}_k = \frac{b^{1-\xi(n)}a^{\xi(n)}\mathfrak{F}_n + a^{1-\xi(n)}b^{\xi(n)}\mathfrak{F}_{n-1} - iab - a(1-iab)}{ab},$$

$$\sum_{k=0}^{n-1} \mathfrak{L}_k = \frac{b^{\xi(n)}a^{1-\xi(n)}\mathfrak{L}_n + b^{1-\xi(n)}a^{\xi(n)}\mathfrak{L}_{n-1} + 1 - 2i - iab + a(-2+iab)}{ab}.$$

Proof. For proof we use Binet formula. If n is even, then

$$\sum_{k=0}^{n-1} \mathfrak{F}_k = \sum_{k=0}^{\frac{n-2}{2}} \mathfrak{F}_{2k+1} + \sum_{k=0}^{\frac{n-2}{2}} \mathfrak{F}_{2k}$$

$$= \frac{1}{\alpha-\beta}\left(\sum_{k=0}^{\frac{n-2}{2}} \frac{(1-i\,\beta)\alpha^{2k+1} - (1-i\alpha)\beta^{2k+1}}{(ab)^k} + a\sum_{k=0}^{\frac{n-2}{2}} \frac{(1-i\,\beta)\alpha^{2k} - (1-i\alpha)\beta^{2k}}{(ab)^k} \right)$$

$$= \frac{1}{\alpha-\beta}\left(\frac{(1-i\,\beta)(\alpha^{n+1}-\alpha(ab)^{\frac{n}{2}})(\beta^2-ab) - (1-i\,\alpha)(\beta^{n+1}-\beta(ab)^{\frac{n}{2}})(\alpha^2-ab)}{(ab)^{\frac{n}{2}-1}(\alpha^2-ab)(\beta^2-ab)} \right)$$
$$+ \frac{a}{\alpha-\beta}\left(\frac{(1-i\,\beta)(\alpha^n-(ab)^{\frac{n}{2}})(\beta^2-ab) - (1-i\,\alpha)(\beta^n-(ab)^{\frac{n}{2}})(\alpha^2-ab)}{-(ab)^{\frac{n}{2}-1}(ab)^3} \right)$$

$$= \frac{1}{\alpha-\beta}\left(\frac{\begin{array}{c}(ab)^2((1-i\,\beta)\alpha^{n-1} - (1-i\,\alpha)\beta^{n-1}) - ab((1-i\,\beta)\alpha^{n+1} - (1-i\,\alpha)\beta^{n+1}) \\ +(ab)^{\frac{n}{2}+1}(\alpha(1-i\,\beta) - (1-i\,\alpha)\beta) - (ab)^{\frac{n}{2}}(\alpha(1-i\,\alpha) - (1-i\beta)\beta)\end{array}}{-(ab)^{\frac{n}{2}+2}} \right)$$
$$+ \frac{a}{\alpha-\beta}\left(\frac{\begin{array}{c}(ab)^2((1-i\,\beta)\alpha^{n-2} - (1-i\,\alpha)\beta^{n-2}) - ab((1-i\,\beta)\alpha^{n} - (1-i\,\alpha)\beta^{n}) \\ +(ab)^{\frac{n}{2}+1}(-1+\mathrm{i}\alpha+1-i\beta) - (ab)^{\frac{n}{2}+1}(\alpha^2(1-i\,\alpha) - (1-i\beta)\beta^2)\end{array}}{-(ab)^{\frac{n}{2}+2}} \right)$$

$$= \frac{1}{-ab}\left[\mathfrak{F}_{n-1} - \mathfrak{F}_{n+1} + iab + \mathfrak{F}_{n-2} - \mathfrak{F}_n - ai(1+ab)\right]$$

$$= \frac{1}{ab}\left[a\mathfrak{F}_{n-1} + b\mathfrak{F}_n - iab - a(1-iab)\right].$$

If n is odd, then

$$\sum_{k=0}^{n-1} \mathfrak{F}_k = \sum_{k=0}^{\frac{n-3}{2}} \mathfrak{F}_{2k+1} + \sum_{k=0}^{\frac{n-1}{2}} \mathfrak{F}_{2k}$$

$$= \frac{1}{\alpha - \beta}\left(\sum_{k=0}^{\frac{n-3}{2}} \frac{(1-i\,\beta)\alpha^{2k+1} - (1-i\alpha)\beta^{2k+1}}{(ab)^k} + a\sum_{k=0}^{\frac{n-1}{2}} \frac{(1-i\,\beta)\alpha^{2k} - (1-i\alpha)\beta^{2k}}{(ab)^k}\right)$$

$$= \frac{1}{\alpha - \beta}\left(\frac{(1-i\,\beta)(\alpha^n - \alpha(ab)^{\frac{n-1}{2}})(\beta^2 - ab) - (1-i\,\alpha)(\beta^n - \beta(ab)^{\frac{n-1}{2}})(\alpha^2 - ab)}{(ab)^{\frac{n-3}{2}}(\alpha^2 - ab)(\beta^2 - ab)}\right)$$
$$+ \frac{a}{\alpha - \beta}\left(\frac{(1-i\,\beta)(\alpha^{n+1} - (ab)^{\frac{n+1}{2}})(\beta^2 - ab) - (1-i\,\alpha)(\beta^{n+1} - (ab)^{\frac{n+1}{2}})(\alpha^2 - ab)}{(ab)^{\frac{n-1}{2}}(\alpha^2 - ab)(\beta^2 - ab)}\right)$$

$$= \frac{1}{\alpha - \beta}\left(\frac{\begin{array}{c}(ab)^2((1-i\,\beta)\alpha^{n-2} - (1-i\,\alpha)\beta^{n-2}) - ab((1-i\,\beta)\alpha^n - (1-i\,\alpha)\beta^n) \\ +(ab)^{\frac{n+1}{2}}(\alpha^2 - \beta^2 + (\beta^3 - \alpha^3)i) + (ab)^{\frac{n+3}{2}}(\alpha - \beta)i\end{array}}{-(ab)^{\frac{n+3}{2}}}\right)$$
$$+ \frac{a}{\alpha - \beta}\left(\frac{\begin{array}{c}(ab)^2((1-i\,\beta)\alpha^{n-1} - (1-i\,\alpha)\beta^{n-1}) - ab((1-i\,\beta)\alpha^{n+1} - (1-i\,\alpha)\beta^{n+1}) \\ +(ab)^{\frac{n+1}{2}}(\beta - \beta^2 i - \alpha + \alpha^2 i) + (ab)^{\frac{n+1}{2}}(\alpha - \beta)i\end{array}}{-(ab)^{\frac{n+5}{2}}}\right)$$

$$= \frac{1}{-ab}[\mathfrak{F}_{n-1} - \mathfrak{F}_{n+1} + iab + \mathfrak{F}_{n-2} - \mathfrak{F}_n + a(1 - iab)]$$
$$= \frac{1}{ab}[a\mathfrak{F}_n + b\mathfrak{F}_{n-1} - iab - a(1 - iab)]$$

If we combine the results we get the desired result.

Conclusion

In this manuscript the gaussian bi-periodic Fibonacci and the gaussian bi-periodic Lucas sequences are defined on the complex plane. The basic properties of these sequences are obtained such as Binet formulas, generating functions, D'ocagne formulas, Catalan identitie... Simple formulas for the sum of the first n th elements of the sequences are denoted.

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